LoopFest X
Evanston IL, May 12–14, 2011

Understanding and Proving the Expansion by Regions

Bernd Jantzen

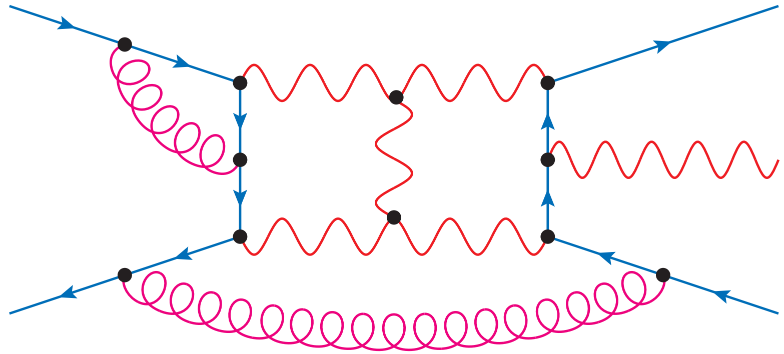
RWTH Aachen University

Disclaimer: This talk is <u>not</u> a practical user's guide on learning how to expand by regions, but a demonstration of the method's correctness.

I The strategy of regions

Starting point: (multi-)loop integral

[no effective theory required]



$$F = \int d^d k_1 \int d^d k_2 \cdots \frac{1}{(k_1+p_1)^2 - m_1^2} \times$$
$$\times \frac{1}{(k_1+k_2+p_2)^2 - m_2^2} \cdots$$

- complicated function of internal masses m_i and kinematical parameters p_i^2, $p_i \cdot p_j$
- exact evaluation often hard or impossible

Exploit parameter hierarchies, e.g. large energies $Q \gg$ small masses m:

$\hookrightarrow$ expand integral in small ratios $\frac{m^2}{Q^2}$

$\hookrightarrow$ simplification achieved if expansion of integrand before integration

But:

⋆ loop-momentum components k_i^μ can take any values (large, small, mixed, ...)

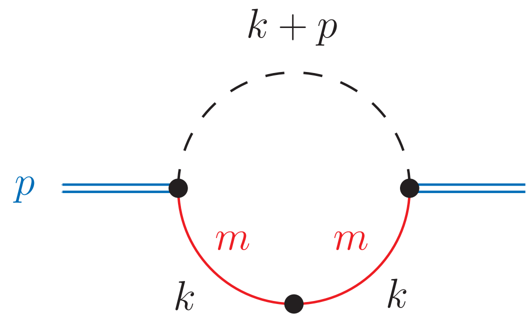
⋆ naive expansions of integrand may generate new singularities

$\hookrightarrow$ Need sophisticated methods of asymptotic expansions.

Simple example: large-momentum expansion

$$F = \int \frac{Dk}{(k+p)^2\,(k^2-m^2)^2}$$

$$\left[\int Dk \equiv \mu^{2\epsilon} e^{\epsilon\gamma_E} \int \frac{d^dk}{i\pi^{d/2}} \qquad d = 4-2\epsilon\right]$$



Large momentum $|p^2| \gg m^2$ $\rightsquigarrow$ expand in $\frac{m^2}{p^2}$.

Integral is UV- and IR-finite, the exact result is known: $[p^2 \to p^2 + i0]$

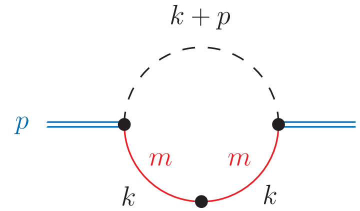
$$F = \frac{1}{p^2}\left[\ln\left(\frac{-p^2}{m^2}\right) + \ln\left(1-\frac{m^2}{p^2}\right)\right] + \mathcal{O}(\epsilon)$$

$$\xrightarrow[\text{expand}]{} \frac{1}{p^2}\left[\ln\left(\frac{-p^2}{m^2}\right) - \sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{m^2}{p^2}\right)^n\right] + \mathcal{O}(\epsilon)$$

Now assume that we could <u>not</u> calculate this integral exactly . . .

Large-momentum expansion (2)

$$F = \int \frac{Dk}{(k+p)^2\,(k^2-m^2)^2}$$



Large momentum $|p^2| \gg m^2$

$\hookrightarrow$ expand integrand before integration:

Expansion by regions

Beneke, Smirnov, Nucl. Phys. B 522, 321 (1998)
Smirnov, Rakhmetov, Theor. Math. Phys. 120, 870 (1999)
Smirnov, Phys. Lett. B 465, 226 (1999)

$\hookrightarrow$ here 2 relevant **regions**:

[originating from integral, not d.o.f. in effective theory]

- **hard** (h): $k \sim p \Rightarrow \sum_i T_i^{(h)} \frac{1}{(k^2-m^2)^2} = \sum_{i=0}^{\infty} (1+i) \frac{(m^2)^i}{(k^2)^{2+i}}$
- **soft** (s): $k \sim m \Rightarrow \sum_j T_j^{(s)} \frac{1}{(k+p)^2} = \sum_{j_1,j_2=0}^{\infty} \frac{(j_1+j_2)!}{j_1!\,j_2!} \frac{(-2k\cdot p)^{j_1}\,(-k^2)^{j_2}}{(p^2)^{1+j_1+j_2}}$

$\Rightarrow$ Integrate each expanded term over the **whole integration domain**.

$\Rightarrow$ Set scaleless integrals to zero ($\rightsquigarrow$ like in dimensional regularization).

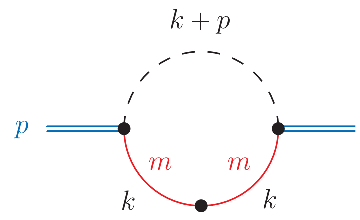
Leading-order contributions:

- **hard:** $F_0^{(h)} = \int \frac{Dk}{(k+p)^2\,(k^2)^2} = \frac{1}{p^2}\left(-\frac{1}{\epsilon} + \mathcal{O}(\epsilon)\right)\left(\frac{\mu^2}{-p^2}\right)^{\epsilon} \rightsquigarrow$ IR-singular!
- **soft:** $F_0^{(s)} = \int \frac{Dk}{p^2\,(k^2-m^2)^2} = \frac{1}{p^2}\left(\frac{1}{\epsilon} + \mathcal{O}(\epsilon)\right)\left(\frac{\mu^2}{m^2}\right)^{\epsilon} \rightsquigarrow$ UV-singular!

$\hookrightarrow$ Contributions are manifestly homogeneous in the expansion parameter $\frac{m^2}{p^2}$.

Large-momentum expansion (3)

$$F = \int \frac{Dk}{(k+p)^2\,(k^2-m^2)^2}$$



Leading-order contributions:

- **hard:** $F_0^{(h)} = \frac{1}{p^2}\left[-\frac{1}{\epsilon} + \ln\left(\frac{-p^2}{\mu^2}\right)\right] + \mathcal{O}(\epsilon) \rightsquigarrow$ IR-singular!
- **soft:** $F_0^{(s)} = \frac{1}{p^2}\left[\frac{1}{\epsilon} + \ln\left(\frac{\mu^2}{m^2}\right)\right] + \mathcal{O}(\epsilon) \rightsquigarrow$ UV-singular!

↪ Singularities are cancelled in the sum of all contributions, exact result approximated:

$$F_0 = F_0^{(h)} + F_0^{(s)} = \frac{1}{p^2}\ln\left(\frac{-p^2}{m^2}\right) + \mathcal{O}(\epsilon) = F + \mathcal{O}\left(\frac{m^2}{(p^2)^2}\right) \quad \checkmark$$

Expand to all orders in $\frac{m^2}{p^2}$: $[(\alpha)_n = \Gamma(\alpha+n)/\Gamma(\alpha)]$

$$F^{(h)} = \frac{1}{p^2}\,\frac{e^{\epsilon\gamma_E}\,\Gamma(1+\epsilon)\,\Gamma^2(1-\epsilon)}{(-\epsilon)\,\Gamma(1-2\epsilon)}\left(\frac{\mu^2}{-p^2}\right)^{\epsilon}\sum_{i=0}^{\infty}\frac{(2\epsilon)_i}{i!}\left(\frac{m^2}{p^2}\right)^{i} = F_0^{(h)} + \frac{2}{p^2}\ln\left(1-\frac{m^2}{p^2}\right) + \mathcal{O}(\epsilon)$$

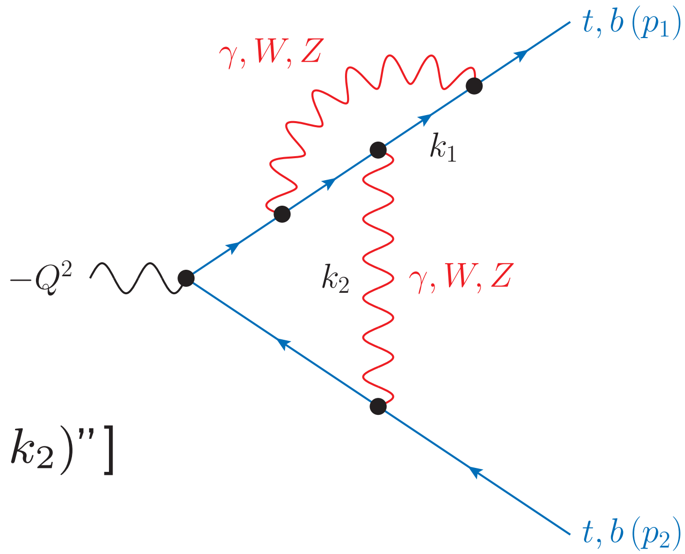
$$F^{(s)} = \frac{1}{p^2}\,e^{\epsilon\gamma_E}\,\Gamma(\epsilon)\left(\frac{\mu^2}{m^2}\right)^{\epsilon}\sum_{j=0}^{\infty}\frac{(\epsilon)_j}{(1-\epsilon)_j}\left(\frac{m^2}{p^2}\right)^{j} = F_0^{(s)} - \frac{1}{p^2}\ln\left(1-\frac{m^2}{p^2}\right) + \mathcal{O}(\epsilon)$$

$$\hookrightarrow F = F^{(h)} + F^{(s)} = \frac{1}{p^2}\left[\ln\left(\frac{-p^2}{m^2}\right) + \ln\left(1-\frac{m^2}{p^2}\right)\right] + \mathcal{O}(\epsilon) \quad \checkmark$$

⇒ Full result F exactly reproduced.

"Real-life" example

The expansion by regions has been applied to many complicated loop integrals.



Example: Denner, B.J., Pozzorini '08

2-loop vertex integral in the high-energy limit

$Q^2 \gg m_t^2$ $\rightsquigarrow$ 9 relevant regions: [labelled "$(k_1 - k_2)$"]

$(h-h)$, $(1c-h)$, $(h-2c)$, $(1c-1c)$, $(1c-2c)$,
$(2c-2c)$, $(us-2c)$, $(1c-2uc)$, $(2uc-2uc)$

$\hookrightarrow$ Next-to-leading-logarithmic result obtained and cross-checked with other methods.

Questions: Why does this expansion by regions work?

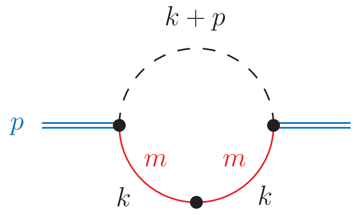
- Large-momentum example: Didn't we double-count every $k \in \mathbb{R}^d$ when replacing $\int Dk \to \int Dk\, T_0^{(h)} + \int Dk\, T_0^{(s)}$?

- What ensures the cancellation of singularities? (IR $\leftrightarrow$ UV!)

- How do we know that the chosen set of regions is complete?

II Why does the method work?

Idea based on a 1-dimensional example from M. Beneke in the book
Smirnov, *Applied Asymptotic Expansions In Momenta And Masses*



Back to the large-momentum example

Let us show step by step how the expansions reproduce the full result.

The **expansions** $\sum_i T_i^{(h)}$, $\sum_j T_j^{(s)}$ **converge absolutely** within **domains** D_h, D_s:

$$(h):\ \frac{1}{(k^2-m^2)^2} = \sum_i T_i^{(h)} \frac{1}{(k^2-m^2)^2} \text{ within } D_h = \left\{k \in \mathbb{R}^d \middle| |k^2| \geq \Lambda^2\right\},$$

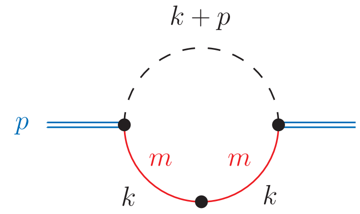
$$(s):\ \frac{1}{(k+p)^2} = \sum_j T_j^{(s)} \frac{1}{(k+p)^2} \text{ within } D_s = \left\{k \in \mathbb{R}^d \middle| |k^2| < \Lambda^2\right\},$$

with $m^2 \ll \Lambda^2 \ll |p^2| \rightsquigarrow D_h \cup D_s = \mathbb{R}^d,\ D_h \cap D_s = \emptyset$.

The expansions **commute** with integrals restricted to the corresponding domains:

$$F = \int\limits_{k\in\mathbb{R}^d} Dk \underbrace{\frac{1}{(k+p)^2\,(k^2-m^2)^2}}_{I} = \sum_i \int\limits_{k\in D_h} Dk\, T_i^{(h)} I + \sum_j \int\limits_{k\in D_s} Dk\, T_j^{(s)} I$$

Continue transforming the expression for the full integral:



$$F = \int\limits_{k\in\mathbb{R}^d} Dk \underbrace{\frac{1}{(k+p)^2\,(k^2-m^2)^2}}_{I} = \sum_i \int\limits_{k\in D_h} Dk\, T_i^{(h)} I + \sum_j \int\limits_{k\in D_s} Dk\, T_j^{(s)} I$$

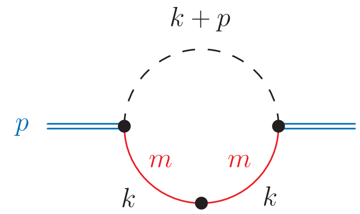
$$= \sum_i \Bigg(\int\limits_{k\in\mathbb{R}^d} Dk\, T_i^{(h)} I - \sum_j \int\limits_{k\in D_s} Dk\, T_j^{(s)} T_i^{(h)} I \Bigg) + \sum_j \Bigg(\int\limits_{k\in\mathbb{R}^d} Dk\, T_j^{(s)} I - \sum_i \int\limits_{k\in D_h} Dk\, T_i^{(h)} T_j^{(s)} I \Bigg)$$

The expansions commute: $\boxed{T_i^{(h)} T_j^{(s)} I = T_j^{(s)} T_i^{(h)} I \equiv T_{i,j}^{(h,s)} I}$

$\Rightarrow$ **Identity:** $$F = \underbrace{\sum_i \int Dk\, T_i^{(h)} I}_{\boldsymbol{F^{(h)}}} + \underbrace{\sum_j \int Dk\, T_j^{(s)} I}_{\boldsymbol{F^{(s)}}} - \underbrace{\sum_{i,j} \int Dk\, T_{i,j}^{(h,s)} I}_{\boldsymbol{F^{(h,s)}}}$$

All terms integrated over the whole integration domain $\mathbb{R}^d$ as prescribed for the expanding by regions $\Rightarrow$ location of boundary Λ between D_h, D_s irrelevant.

Identity: $F = \underbrace{\sum_i \int Dk\, T_i^{(h)} I}_{\boldsymbol{F^{(h)}}} + \underbrace{\sum_j \int Dk\, T_j^{(s)} I}_{\boldsymbol{F^{(s)}}} - \underbrace{\sum_{i,j} \int Dk\, T_{i,j}^{(h,s)} I}_{\boldsymbol{F^{(h,s)}}}$



Additional overlap contribution $\boldsymbol{F^{(h,s)}}$?

$$F^{(h,s)} = \sum_{i=0}^{\infty} (1+i) \sum_{j_1,j_2=0}^{\infty} (-1)^{j_2} \frac{(j_1+j_2)!}{j_1!\, j_2!} \frac{(m^2)^i}{(p^2)^{1+j_1+j_2}} \int Dk \frac{(-2k\cdot p)^{j_1}}{(k^2)^{2+i-j_2}} = 0 \quad \text{scaleless!}$$

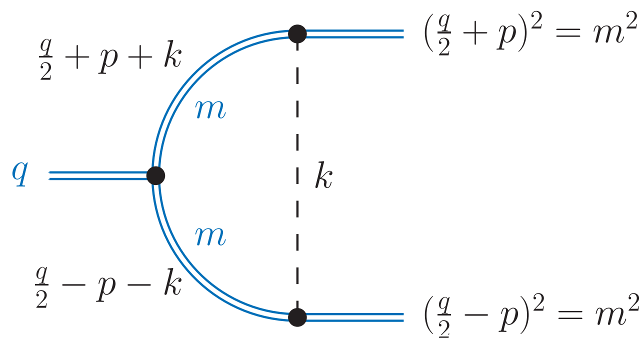
[Actually $\int \frac{Dk}{(k^2)^2} = \frac{1}{\epsilon_{\mathsf{UV}}} - \frac{1}{\epsilon_{\mathsf{IR}}}$ cancels corresponding singularities in $F^{(h)}$ and $F^{(s)}$.]

$\hookrightarrow$ $\boxed{\boldsymbol{F = F^{(h)} + F^{(s)}}}$ as found before.

But now this identity has been obtained without evaluating F, $F^{(h)}$, $F^{(s)}$!

Example with 3 regions: threshold expansion for heavy-particle pair production

Regions analyzed in Beneke, Smirnov, NPB 522, 321 (1998)



Center-of-mass system: $(q^\mu) = (q_0, \vec{0})$, $(p^\mu) = (0, \vec{p})$

Close to threshold: $q^2 \approx (2m)^2 \Rightarrow$ $q^2 \gg |p^2|$ or $q_0 \gg |\vec{p}|$

$$F = \int \frac{Dk}{(k^2 + q_0 k_0 - 2\vec{p}\cdot\vec{k})\,(k^2 - q_0 k_0 - 2\vec{p}\cdot\vec{k})\,k^2}$$

Relevant regions:

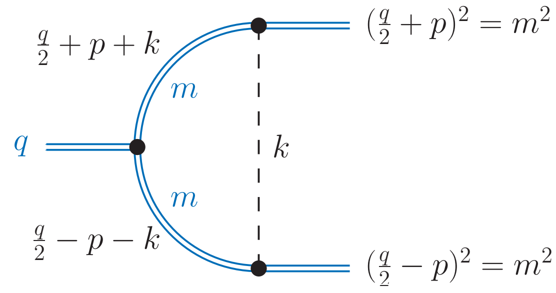
- **hard** (h): $k_0, |\vec{k}| \sim q_0 \Rightarrow$ expand $\sum_j T_j^{(h)}$ in $D_h = \left\{k \in \mathbb{R}^d \middle| |k_0| \gg |\vec{p}| \text{ or } |\vec{k}| \gg |\vec{p}|\right\}$
- **soft** (s): $k_0, |\vec{k}| \sim |\vec{p}| \Rightarrow$ expand $\sum_j T_j^{(s)}$ in $D_s = \left\{k \in \mathbb{R}^d \middle| |\vec{k}| \lesssim |k_0| \lesssim |\vec{p}|\right\}$
- **potential** (p): $k_0 \sim \frac{\vec{p}^{\,2}}{q_0}$, $|\vec{k}| \sim |\vec{p}| \Rightarrow \sum_j T_j^{(p)}$ in $D_p = \left\{k \in \mathbb{R}^d \middle| |k_0| \ll |\vec{k}| \lesssim |\vec{p}|\right\}$

[no explicit boundaries needed]

$\hookrightarrow$ $D_h \cup D_s \cup D_p = \mathbb{R}^d$, $\quad D_h \cap D_s = D_h \cap D_p = D_s \cap D_p = \emptyset$

$\hookrightarrow$ The expansions $T^{(h)}, T^{(s)}, T^{(p)}$ commute with each other.

Threshold expansion (2)



$$F=\int\frac{Dk}{(k^2+q_0k_0-2\vec{p}\cdot\vec{k})\,(k^2-q_0k_0-2\vec{p}\cdot\vec{k})\,k^2}$$

Similar transformations as for the large-momentum example yield the following **identity:**

$$F=F^{(h)}+\underbrace{F^{(s)}}_{=0}+F^{(p)}-\Big(\underbrace{F^{(h,s)}}_{=0}+\underbrace{F^{(h,p)}}_{=0}+\underbrace{F^{(s,p)}}_{=0}\Big)+\underbrace{F^{(h,s,p)}}_{=0\text{ (scaleless)}}$$

with

$$F^{(h)}=-\frac{2e^{\epsilon\gamma_E}\,\Gamma(\epsilon)}{q^2}\left(\frac{4\mu^2}{q^2}\right)^{\epsilon}\sum_{j=0}^{\infty}\frac{(1+\epsilon)_j}{j!\,(1+2\epsilon+2j)}\left(-\frac{4p^2}{q^2}\right)^{j}$$

$$F^{(p)}=\frac{e^{\epsilon\gamma_E}\,\Gamma(\frac{1}{2}+\epsilon)\,\sqrt{\pi}}{2\epsilon\,\sqrt{q^2\,(p^2-i0)}}\left(\frac{\mu^2}{p^2-i0}\right)^{\epsilon}\qquad\text{[higher orders are scaleless]}$$

Exact result reproduced:

$$F^{(h)}+F^{(p)}=F=\frac{e^{\epsilon\gamma_E}\,\Gamma(\epsilon)}{2p^2}\left(\frac{\mu^2}{p^2-i0}\right)^{\epsilon}{}_2F_1\left(\frac{1}{2},1+\epsilon;\frac{3}{2};-\frac{q^2}{4p^2}-i0\right)\quad ✓$$

III The general formalism

The identities obtained for the previous examples are generally valid, under some conditions:

Consider

- a (multiple) integral $F = \int Dk\, I$ over the domain D (e.g. $D = \mathbb{R}^d$),
- a set of N regions $R = \{x_1, \ldots, x_N\}$,
- for each region $x \in R$ an expansion $T^{(x)} = \sum_j T_j^{(x)}$ which converges absolutely in the domain $D_x \subset D$.

Conditions:

- $\bigcup_{x \in R} D_x = D, \qquad D_x \cap D_{x'} = \emptyset \, \forall x \neq x'$
- the expansions commute: $T^{(x)} T^{(x')} I = T^{(x')} T^{(x)} I \equiv T^{(x,x')} I$
- $\exists$ regularization for singularities, e.g. dimensional (+ analytic) reg.

$\Rightarrow$ The integral expression can be transformed as in the previous examples.

The general formalism (2)

Under the above conditions, the following **identity** holds:

$$F = \sum_{x'_1 \in R} F^{(x'_1)} - \sum_{\{x'_1, x'_2\} \subset R} F^{(x'_1,x'_2)} + \ldots - (-1)^n \sum_{\{x'_1, \ldots, x'_n\} \subset R} F^{(x'_1,\ldots,x'_n)} + \ldots - (-1)^N F^{(x_1,\ldots,x_N)}$$

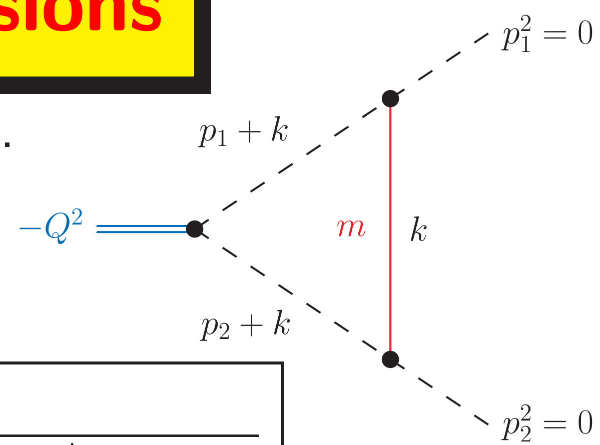
$$\left[F^{(x,\ldots)} \equiv \textstyle\sum_{j,\ldots} \int Dk\, T^{(x,\ldots)}_{j,\ldots} I\right]$$

Comments

- This identity is exact when the expansions are summed to all orders. ✓
- It is independent of the regularization (dim. reg., analytic reg., cut-off, infinitesimal masses/off-shellness, . . .) as long as all individual terms are well-defined.
- Usually regions & regularization are chosen such that multiple expansions $F^{(x'_1,\ldots,x'_n)}$ $(n \geq 2)$ are scaleless and vanish.
  [OK if each $F^{(x)}$ is a *homogeneous* function of the expansion parameter with *unique scaling*.]
- If $\exists\, F^{(x'_1,x'_2,\ldots)} \neq 0 \rightsquigarrow$ relevant overlap contributions ($\to$ "zero-bin subtractions").
  They appear e.g. when avoiding analytic regularization in SCET.

e.g. Manohar, Stewart '06;
Chiu, Fuhrer, Hoang, Kelley, Manohar '09;
. . .

IV Non-commuting expansions



Cannot always choose expansions which commute with each other.

Example: Sudakov form factor

Sudakov limit: $-(p_1 - p_2)^2 = \boxed{Q^2 \gg m^2}$

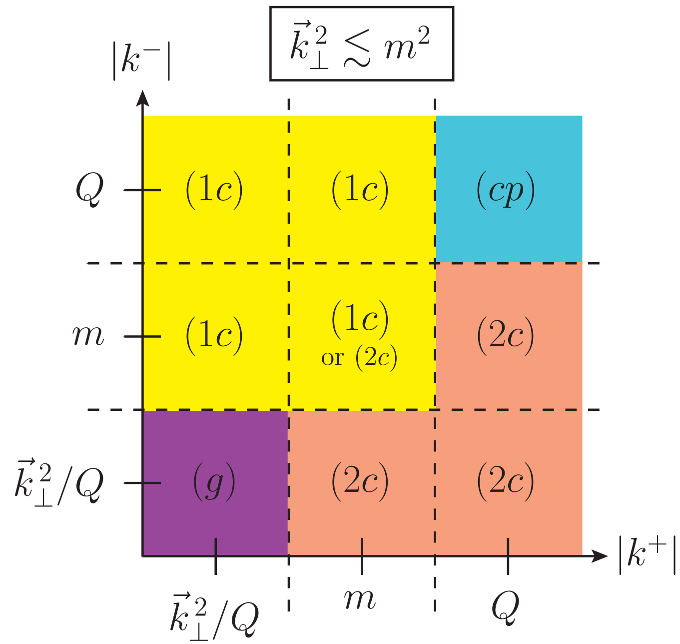
$$F = \int \frac{Dk}{(k^+k^- - \vec{k}_\perp^2 + Qk^+)^{1+\delta}\,(k^+k^- - \vec{k}_\perp^2 + Qk^-)^{1-\delta}\,(k^+k^- - \vec{k}_\perp^2 - m^2)}$$

$\hookrightarrow$ analytic regulator $\delta \to 0$ [light-cone coordinates: $2p_{1,2} \cdot k = Qk^\pm$, $p_{1,2} \cdot k_\perp = 0$]

Regions & domains:

- **hard** (h): $k^+, k^-, |\vec{k}_\perp| \sim Q \Rightarrow D_h = \left\{ k \in \mathbb{R}^d \middle| \vec{k}_\perp^2 \gg m^2 \right\}$
- **1-collinear** $(1c)$: $k^+ \sim \frac{m^2}{Q}$, $k^- \sim Q$, $|\vec{k}_\perp| \sim m$
- **2-collinear** $(2c)$: $k^+ \sim Q$, $k^- \sim \frac{m^2}{Q}$, $|\vec{k}_\perp| \sim m$
- **Glauber** (g): $k^+, k^- \sim \frac{m^2}{Q}$, $|\vec{k}_\perp| \sim m$
- **collinear plane** (cp): $k^+, k^- \sim Q$, $|\vec{k}_\perp| \sim m$
  $\hookrightarrow$ "artificial" region to ensure $\cup_x D_x = \mathbb{R}^d$

[No soft region needed: $T^{(s)} \equiv T^{(1c)} T^{(2c)}$]



Most expansions commute, but $T^{(g)} T^{(cp)} \neq T^{(cp)} T^{(g)}$!

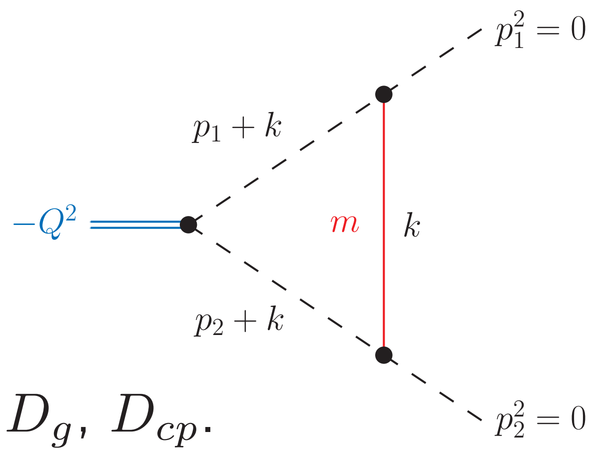
Sudakov form factor (2)

$T^{(g)}T^{(cp)} \neq T^{(cp)}T^{(g)} \rightsquigarrow$ Construct **identity** avoiding combination of (g) and (cp):

$$\begin{aligned} F = {} & F^{(h)} + F^{(1c)} + F^{(2c)} + F^{(g)} + F^{(cp)} \\ & - \Big(F^{(h,1c)} + F^{(h,2c)} + F^{(h,g)} + F^{(h,cp)} + F^{(1c,2c)} + F^{(1c,g)} + F^{(1c,cp)} + F^{(2c,g)} + F^{(2c,cp)} \Big) \\ & + F^{(h,1c,2c)} + F^{(h,1c,g)} + F^{(h,1c,cp)} + F^{(h,2c,g)} + F^{(h,2c,cp)} + F^{(1c,2c,g)} + F^{(1c,2c,cp)} \\ & - \Big(F^{(h,1c,2c,g)} + F^{(h,1c,2c,cp)} \Big) + F^{\text{extra}}_{g\to cp} + F^{\text{extra}}_{cp\to g} \end{aligned}$$

Extra terms:

- $F^{\text{extra}}_{g\to cp}$ involves $T^{(cp)}T^{(g)}$ integrated over $k \in D_{cp}$
- $F^{\text{extra}}_{cp\to g}$ involves $T^{(g)}T^{(cp)}$ integrated over $k \in D_g$



Both extra terms cancel at the integrand level.

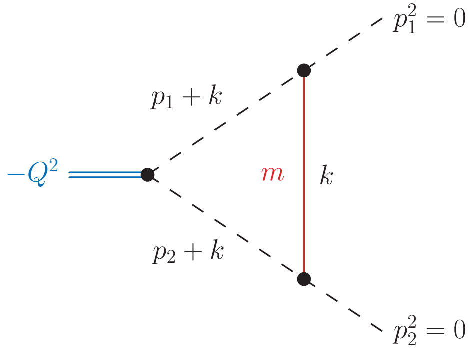
$\hookrightarrow$ They must do so $\rightsquigarrow$ otherwise dependence on boundaries of D_g, D_{cp}.

Usual terms:

- no combination of (g) and (cp)
- all overlap contributions and $F^{(g)}$, $F^{(cp)}$ are scaleless (with analytic regularization)

Sudakov form factor (3)

$$F = \int \frac{Dk}{(k^+k^- - \vec{k}_\perp^2 + Qk^+)^{1+\delta}\,(k^+k^- - \vec{k}_\perp^2 + Qk^-)^{1-\delta}\,(k^+k^- - \vec{k}_\perp^2 - m^2)}$$



Omitting scaleless contributions and vanishing extra terms:

$$F = F^{(h)} + F^{(1c)} + F^{(2c)}$$

Regions explicitly evaluated to all orders in $\frac{m^2}{Q^2}$: [omitting $\mathcal{O}(\delta)$ and $\mathcal{O}(\epsilon)$]

$$F^{(h)} = -\frac{1}{Q^2}\left(\frac{\mu^2}{Q^2}\right)^\epsilon \left\{\frac{1}{\epsilon^2} - \frac{2}{\epsilon}\ln\left(1-\frac{m^2}{Q^2}\right) + \ln^2\left(1-\frac{m^2}{Q^2}\right) - 2\,\mathrm{Li}_2\left(\frac{m^2}{Q^2}\right) - \frac{\pi^2}{12}\right\}$$

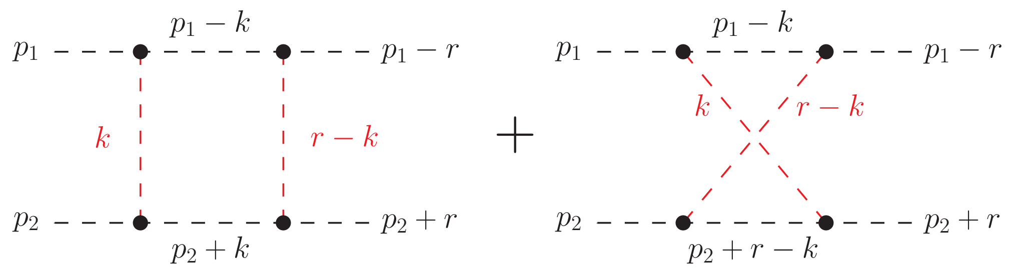
$$F^{(1c)}, F^{(2c)} = -\frac{1}{2Q^2}\left(\frac{\mu^2}{Q^2}\right)^\epsilon \left\{\pm\frac{1}{\delta}\left[\frac{1}{\epsilon} + \ln\frac{Q^2}{m^2} - \ln\left(1-\frac{m^2}{Q^2}\right)\right] - \frac{1}{\epsilon^2} + \frac{2}{\epsilon}\ln\left(1-\frac{m^2}{Q^2}\right)\right.$$
$$\left. + \frac{1}{2}\ln^2\frac{Q^2}{m^2} + \ln\frac{Q^2}{m^2}\ln\left(1-\frac{m^2}{Q^2}\right) - \ln^2\left(1-\frac{m^2}{Q^2}\right) + \mathrm{Li}_2\left(\frac{m^2}{Q^2}\right) + \frac{5}{12}\pi^2\right\}$$

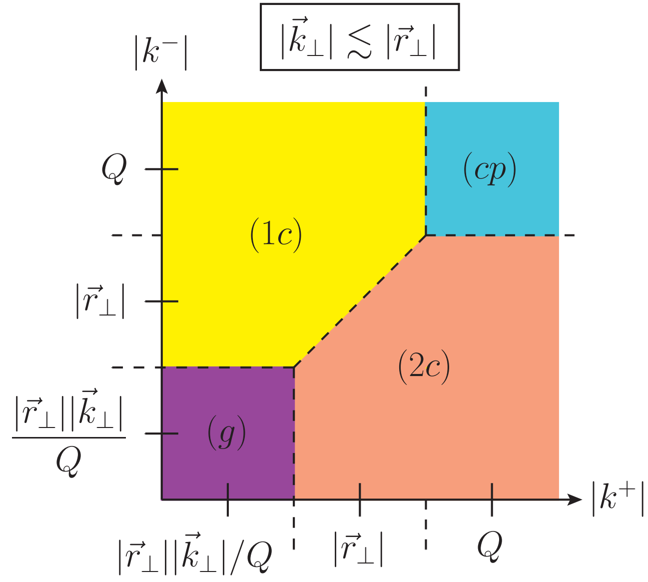
$\hookrightarrow$ $F^{(1c)}$ and $F^{(2c)}$ are not separately finite for $\delta \to 0$, but their sum is.

Compare to exact result:

$$F = -\frac{1}{Q^2}\left\{\frac{1}{2}\ln^2\frac{Q^2}{m^2} + \ln\frac{Q^2}{m^2}\ln\left(1-\frac{m^2}{Q^2}\right) - \mathrm{Li}_2\left(\frac{m^2}{Q^2}\right) + \frac{\pi^2}{3}\right\} \quad \checkmark$$

Last example: forward scattering with small-momentum exchange

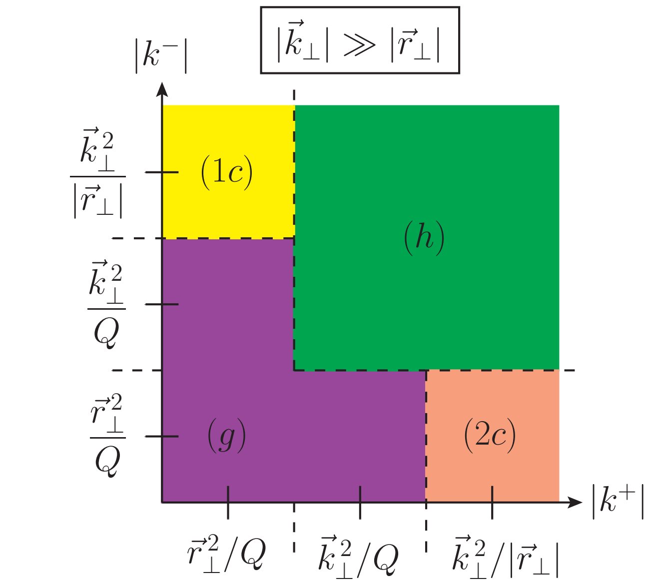




Two light-like particles with large center-of-mass energy exchange a small momentum r:

$$p_1^2 = (p_1 - r)^2 = p_2^2 = (p_2 + r)^2 = 0$$

$$(p_1 + p_2)^2 = \boxed{Q^2 \gg \vec{r}_\perp^{\,2}}\,, \qquad r^\pm \approx \mp \frac{\vec{r}_\perp^{\,2}}{Q}$$



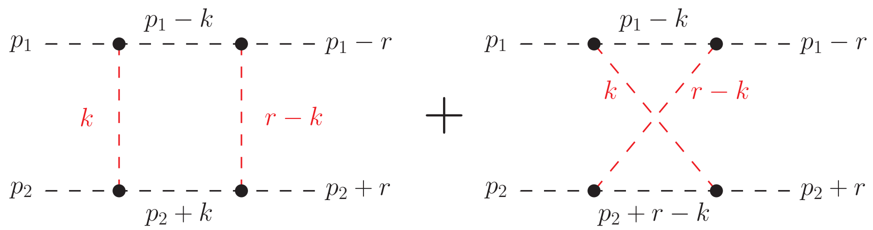
Symmetrize integral under $k \leftrightarrow r - k$
$\hookrightarrow$ avoids divergences at $|k^\pm| \to \infty$ under expansion.

$$F = \frac{1}{2} \int \frac{Dk}{k^2\,(r-k)^2} \left(\frac{1}{\big((p_1-k)^2\big)^{1+\delta}} + \frac{1}{\big((p_1-r+k)^2\big)^{1+\delta}} \right) \times \left(\frac{1}{\big((p_2+k)^2\big)^{1-\delta}} + \frac{1}{\big((p_2+r-k)^2\big)^{1-\delta}} \right)$$

Regions: same as for Sudakov form factor (with scaling $m \to |\vec{r}_\perp|$),

Domains: similar (but more involved for $|\vec{k}_\perp| \gg |\vec{r}_\perp|$)

Forward scattering (2)



Same identity as for Sudakov form factor:

$$\begin{aligned}F &= F^{(h)} + F^{(1c)} + F^{(2c)} + F^{(g)} + F^{(cp)} \\ &\quad - \Big(F^{(h,1c)} + F^{(h,2c)} + F^{(h,g)} + F^{(h,cp)} + F^{(1c,2c)} + F^{(1c,g)} + F^{(1c,cp)} + F^{(2c,g)} + F^{(2c,cp)} \Big) \\ &\quad + F^{(h,1c,2c)} + F^{(h,1c,g)} + F^{(h,1c,cp)} + F^{(h,2c,g)} + F^{(h,2c,cp)} + F^{(1c,2c,g)} + F^{(1c,2c,cp)} \\ &\quad - \Big(F^{(h,1c,2c,g)} + F^{(h,1c,2c,cp)} \Big)\end{aligned}$$

With analytic regulator $\delta \to 0$: $\boxed{F_0 = F_0^{(1c)} + F_0^{(2c)}}$ [$F_0^{(h)}$ suppressed, others scaleless]

$$F_0^{(1c)} = F_0^{(2c)} = \frac{i\pi}{2\, Q^2\, \vec{r}_\perp^{\,2}} \left(\frac{\mu^2}{\vec{r}_\perp^{\,2}} \right)^{\epsilon} \frac{e^{\epsilon\gamma_E}\, \Gamma(1+\epsilon)\, \Gamma^2(-\epsilon)}{\Gamma(-2\epsilon)}$$

Without analytic regularization ($\delta = 0$): [all terms are still well-defined]

$$\boxed{F_0 = F_0^{(1c)} + F_0^{(2c)} + F_0^{(g)} - \Big(F_0^{(1c,2c)} + F_0^{(1c,g)} + F_0^{(2c,g)} \Big) + F_0^{(1c,2c,g)}}$$

$$F_0^{(x,\dots)} = \frac{i\pi}{Q^2\, \vec{r}_\perp^{\,2}} \left(\frac{\mu^2}{\vec{r}_\perp^{\,2}} \right)^{\epsilon} \frac{e^{\epsilon\gamma_E}\, \Gamma(1+\epsilon)\, \Gamma^2(-\epsilon)}{\Gamma(-2\epsilon)} \quad \forall\, x, \ldots \in \{1c, 2c, g\}$$

↪ consistent results independent of regularization: $\frac{1}{2} + \frac{1}{2} = 1 + 1 + 1 - (1 + 1 + 1) + 1$ ✓

↪ agreement with leading-order expansion of full result

V Summary

Expansion by regions for general integrals

- conditions for regions (+ corresponding expansions & domains) established
- identity proven $\rightsquigarrow$ relates exact integral to sum of expanded terms
- this identity includes overlap contributions:

$$F = \sum_{x'_1 \in R} F^{(x'_1)} - \sum_{\{x'_1, x'_2\} \subset R} F^{(x'_1, x'_2)} + \ldots - (-1)^n \sum_{\{x'_1, \ldots, x'_n\} \subset R} F^{(x'_1, \ldots, x'_n)} + \ldots - (-1)^N F^{(x_1, \ldots, x_N)}$$

  $\hookrightarrow$ valid independent of the choice of regularization

- overlap contributions can be scaleless or relevant (depending on regularization)
- successful application to several examples (setup & check of conditions, evaluation of regions to all orders, comparison to exact result)

Non-commuting expansions

- extra terms vanish at the integrand level
- generalized identity without overlap combinations of non-commuting expansions

Extra slides

Practical note: how to find the relevant regions

- Look where the propagators have poles:
  - ⋆ Large-momentum example: $(k+p)^2 = 0$ at $k \sim p$, $k^2 - m^2 = 0$ at $k \sim m$.
  - ⋆ Close the integration contour of one component (e.g. k^0, $k^\pm$).
    For all residues investigate the scaling of the components.

- Use Mellin–Barnes (MB) representations:
  1. Evaluate the full (scalar) integral for general propagator powers n_i in terms of multiple MB integrals.
  2. Close MB contours involving the expansion parameter and extract the leading contributions.
  3. The individual terms can be identified with corresponding regions by their homogeneous scaling with the expansion parameter depending on d and n_i.

  [A subsequent expansion by regions often yields simpler expressions for the contributions.]

- Try all possible regions that you can imagine ...
  If a region does not contribute, its integrals are scaleless.

- When a region is missing, the total result is often (but not always) more singular than it should be. $\rightsquigarrow$ Important cross-check, but no guarantee!