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Evaluation of electroweak two-loop corrections in the high energy limit

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J.H. Kühn, A.A. Penin, *hep-ph/9906545*

J.H. Kühn, A.A. Penin, V.A. Smirnov, *Eur. Phys. J. C 17 (2000) 97*

J.H. Kühn, S. Moch, A.A. Penin, V.A. Smirnov, *Nucl. Phys. B 616 (2001) 286*

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B. Jantzen, J.H. Kühn, A.A. Penin, V.A. Smirnov, *Nucl. Phys. B 731 (2005) 188*

B. Jantzen, V.A. Smirnov, hep-ph/0603133 (→ Eur. Phys. J. C)

I Motivation

Electroweak precision physics

- experimentally measured (LEP, Tevatron) at energy scales $\lesssim M_{W,Z}$
- upcoming accelerators (LHC, ILC) $\to$ TeV region
- new energy domain $\sqrt{s} \gg M_{W,Z}$ becomes accessible!

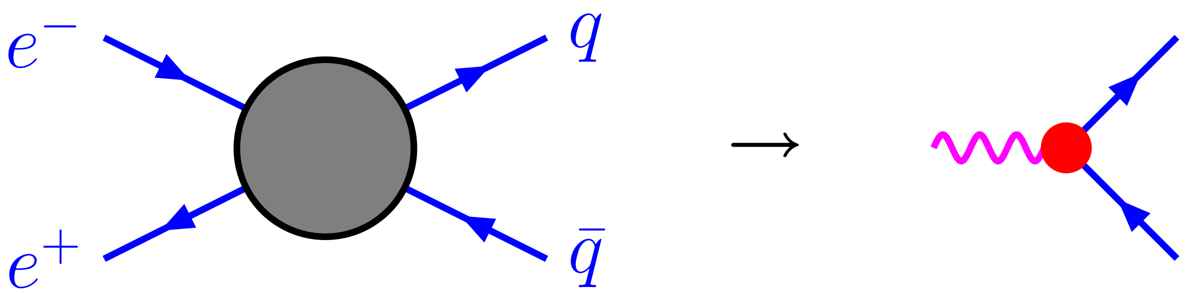
Electroweak radiative corrections at high energies $\sqrt{s} \sim$ TeV $\gg M_{W,Z}$

Fadin et al. '00; Kühn et al. '00, '01, '05; Denner et al. '01, '03, '04, '05; Pozzorini '04; B.J. et al. '03, '04, '05, '06; ...

large corrections in *exclusive* cross sections

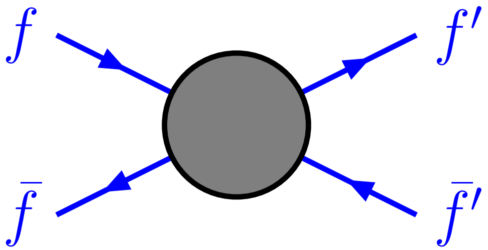
- electroweak corrections with large logarithms: $\alpha^n \ln^j(s/M_{W,Z}^2)$
- leading Sudakov logarithms ($j = 2n$, LL), but large coefficients in front of subleading logarithms ($j \leq 2n-1 \Rightarrow$ NLL, NNLL, N^{3}LL, ...)
- origin of Sudakov logarithms: mass singularities for $M_{W,Z} \to 0$
- 1-loop corrections $\sim$ 10%, 2-loop corrections $\sim$ 1%, needed for ILC
- individual logarithmic contributions even larger, but strong cancellations

II Four-fermion scattering



Four-fermion scattering: $f\bar{f} \to f'\bar{f}'$, important class of processes

Factorization of QED contributions:



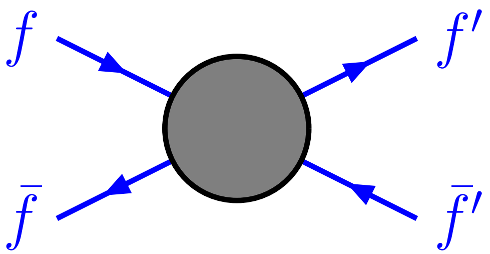
$$A = U_{\text{QED}} \cdot A_{\text{EW}}$$

- QED factor $U_{\text{QED}} \to$ IR singularities from virtual massless photons (regularized dimensionally or by small photon mass, compensated by real corrections)
- amplitude $A_{\text{EW}} \to$ remaining electroweak contributions, IR-safe
- calculate A_{EW} by evaluating A/U_{QED} with $M_{\text{photon}} = M_W$
  $\hookrightarrow$ works at NNLL accuracy ✓

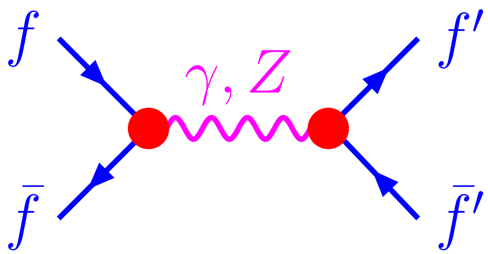
Problem at N^{3}LL (2-loop linear logarithm):
mixing of gauge groups $SU(2) \times U(1)_Y$ through Higgs mechanism
$\Rightarrow$ use simplified model without mixing: B.J., Kühn, Penin, Smirnov '04, '05

- factorization of QED contributions works at N^{3}LL accuracy
- single mass parameter: $M = M_W = M_{Z=W^3} = M_{\text{photon}=B}$
- include mass difference $(M_Z - M_W)$ by expansion around $M_Z \approx M_W$
- remaining error $\sim \mathcal{O}(\sin^2\theta_W) \sim 20\%$ in coefficient of linear 2-loop logarithm

Factorization into form factor and reduced amplitude:

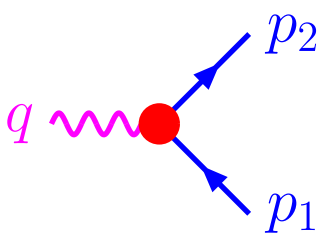


$$A_{\mathsf{EW}} = \frac{ig^2}{s} F^2 \tilde{A}$$



$\times \tilde{A}$

Form factor F of vector current:



$$= F \cdot \bar{u}(p_2)\, \gamma^\mu\, u(p_1) + \mathcal{O}(\text{fermion masses})$$

High energy behaviour $s \sim |t| \sim |u| \gg M_{W,Z}^2$ references: see Kühn et al. '01

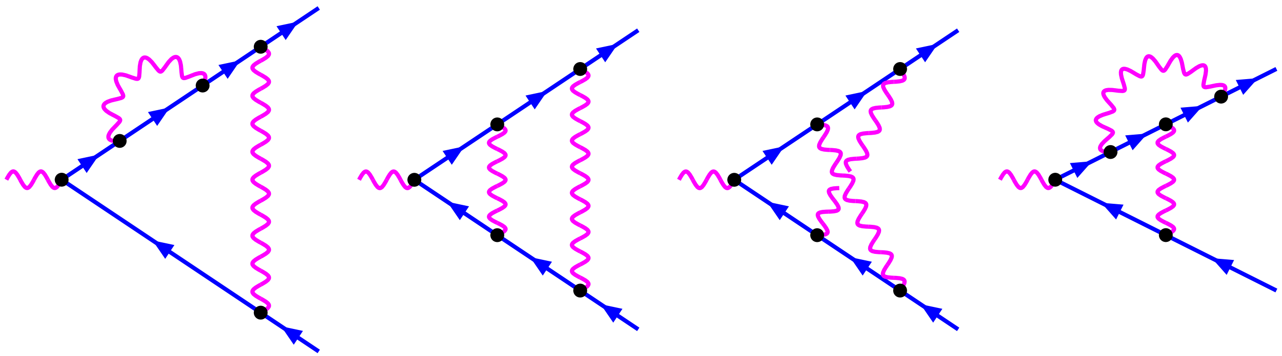
- all double logarithms $\alpha^n \ln^{2n} \rightsquigarrow$ form factors F^2
- *reduced amplitude* $\tilde{A} \longrightarrow$ only single logarithms $\alpha^n \ln^n$
- $\tilde{A}$ can be obtained from 1-loop and massless 2-loop calculations

$\Rightarrow$ For full logarithmic (N^{3}LL) 2-loop amplitude: need form factor F

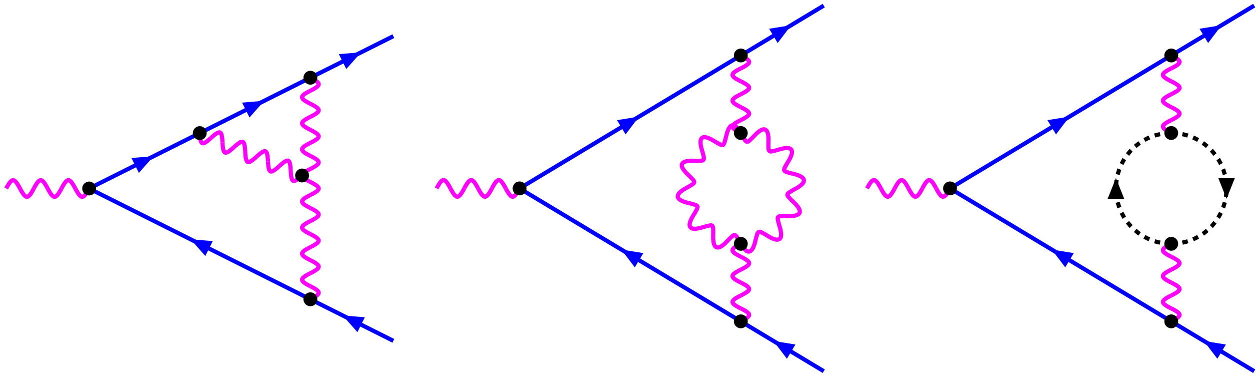
$\hookrightarrow$ evaluate 2-loop vertex diagrams

SU(2) form factor in two loops: diagrams

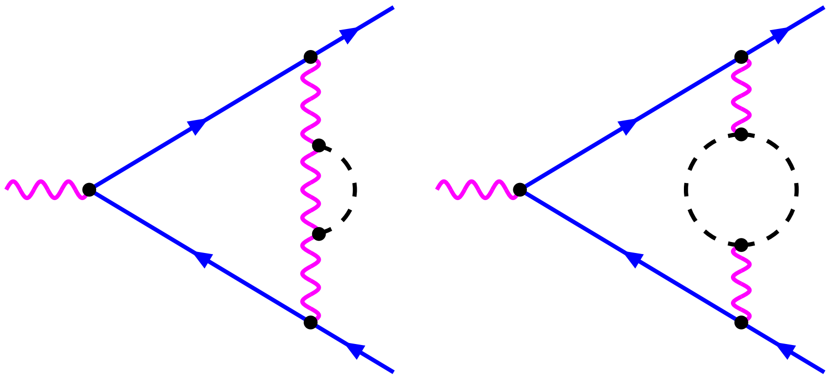
Abelian:
$\hookrightarrow$ U(1)



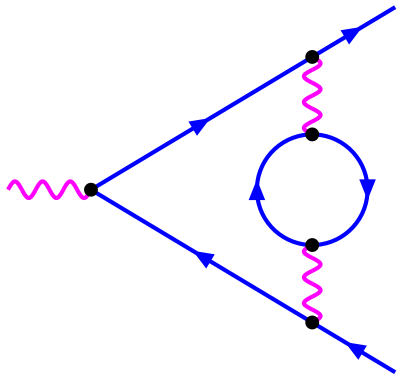
non-Abelian:



Higgs:



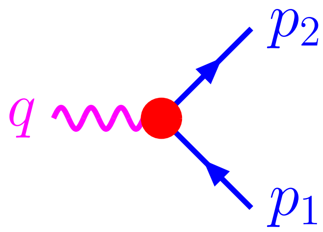
fermion:



+ 1-loop×1-loop corrections + renormalization

High energy behaviour of the form factor

$\hookrightarrow$ Sudakov limit:



$$= F(Q^2) \cdot \bar{u}(p_2)\,\gamma^\mu\,u(p_1)$$

- momentum transfer $-q^2 \equiv Q^2 \gg M^2 \equiv M_{W,Z}^2$

  $\Big[$Euclidean $Q^2 > 0$, real F $\xrightarrow[\text{continuation}]{\text{analytic}}$ Minkowskian $Q^2 = -s - i0 < 0\Big]$

- neglect fermion masses $\to$ external on-shell fermions: $p_1^2 = p_2^2 = 0$

- *logarithmic approximation*: neglect terms suppressed by a factor of M^2/Q^2

  $\hookrightarrow$ works well for 2-loop n_f contribution where the exact result in M^2/Q^2 is known

  B.F., Kühn, Moch '03

  $\Rightarrow$ contains powers of the large logarithm $\ln(Q^2/M^2)$

  $\Rightarrow$ leading order of asymptotic expansion in M^2/Q^2

- only 2-loop logarithms $\ln^{4,3,2,1}$, non-logarithmic constant more difficult

- choose $M_{\text{Higgs}} = M_W$ $\to$ calculation easier, affects only N^{3}LL, small error

SU(2) form factor in two loops: result

$$\left(\frac{\alpha}{4\pi}\right)^2 \left[\frac{9}{32}\ln^4\left(\frac{Q^2}{M^2}\right) - \frac{43}{48}\ln^3\left(\frac{Q^2}{M^2}\right) + \left(\frac{7}{8}\pi^2 - \frac{235}{48}\right)\ln^2\left(\frac{Q^2}{M^2}\right) \right.$$

confirmed ✓

$$\left. + \left(\frac{13}{2}\sqrt{3}\,\mathrm{Cl}_2\left(\frac{\pi}{3}\right) + \frac{15}{4}\sqrt{3}\pi - \frac{61}{2}\zeta_3 - \frac{11}{24}\pi^2 + \frac{65}{4}\right)\ln\left(\frac{Q^2}{M^2}\right) \right]$$

new!

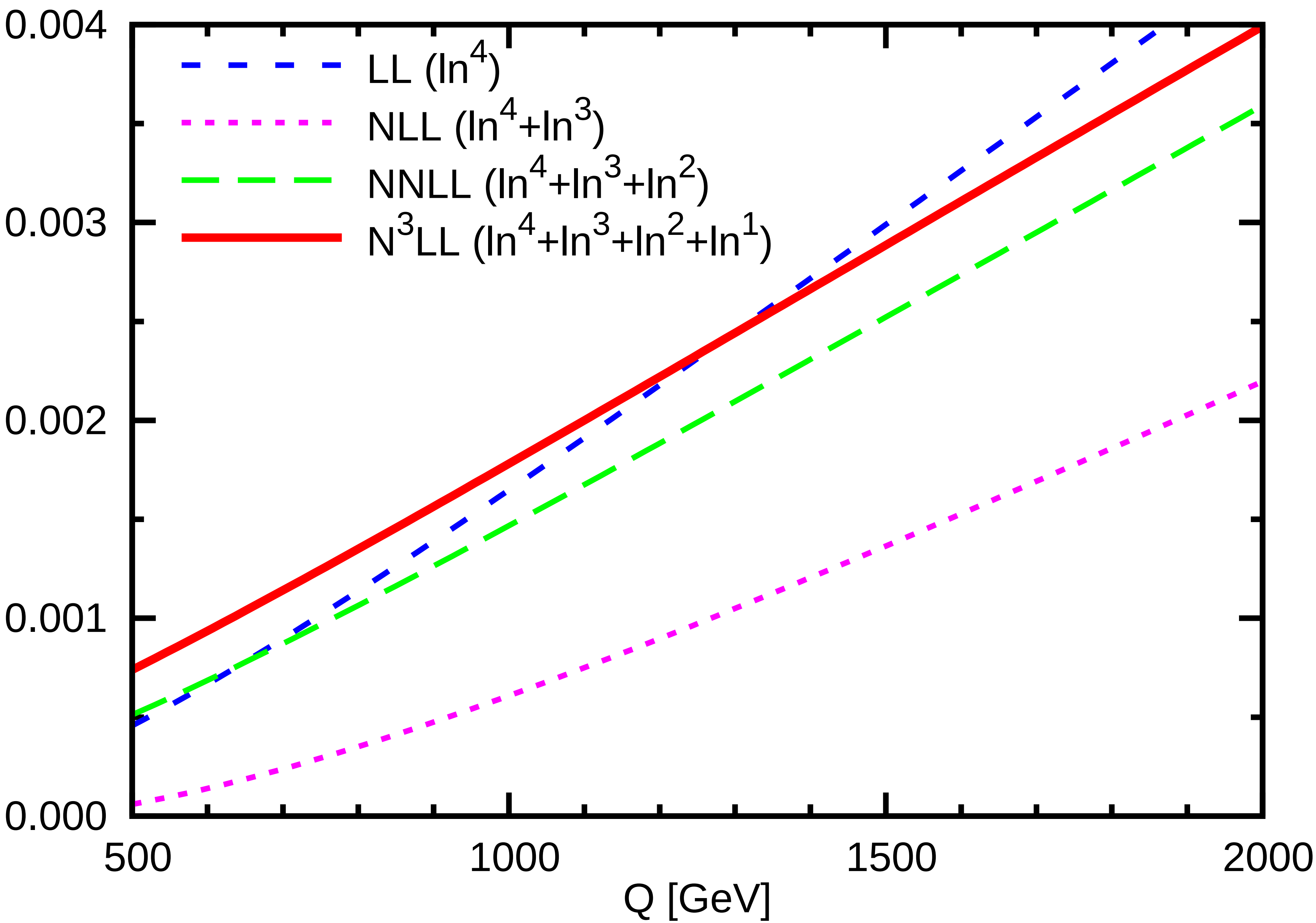
B.J., Kühn, Moch '03; B.J., Kühn, Penin, Smirnov '04, '05; B.J., Smirnov '06

$\ln^{4,3,2}$: Kühn, Moch, Penin, Smirnov '01

Sizes of logarithmic contributions (at $Q = 1$ TeV in per mil):

Abelian:	$+\,0.3\ln^4$	$-\,1.7\ln^3$	$+\,8.2\ln^2$	$-\,11\ln$	$+\,15$
	$+1.6$	-2.0	$+1.9$	-0.5	$+0.1$
fermionic:		$-\,1.0\ln^3$	$+\,9.5\ln^2$	$-\,26\ln$	$+\,42$
		-1.2	$+2.2$	-1.2	$+0.4$
non-Abelian + Higgs:		$+\,1.8\ln^3$	$-\,14\ln^2$	$+\,43\ln$	$-\,\ldots$
		$+2.1$	-3.2	$+2.0$	
total:	$+\,0.3\ln^4$	$-\,0.9\ln^3$	$+\,3.7\ln^2$	$+\,6.9\ln$	
	$+1.6$	-1.0	$+0.9$	$+0.3$	

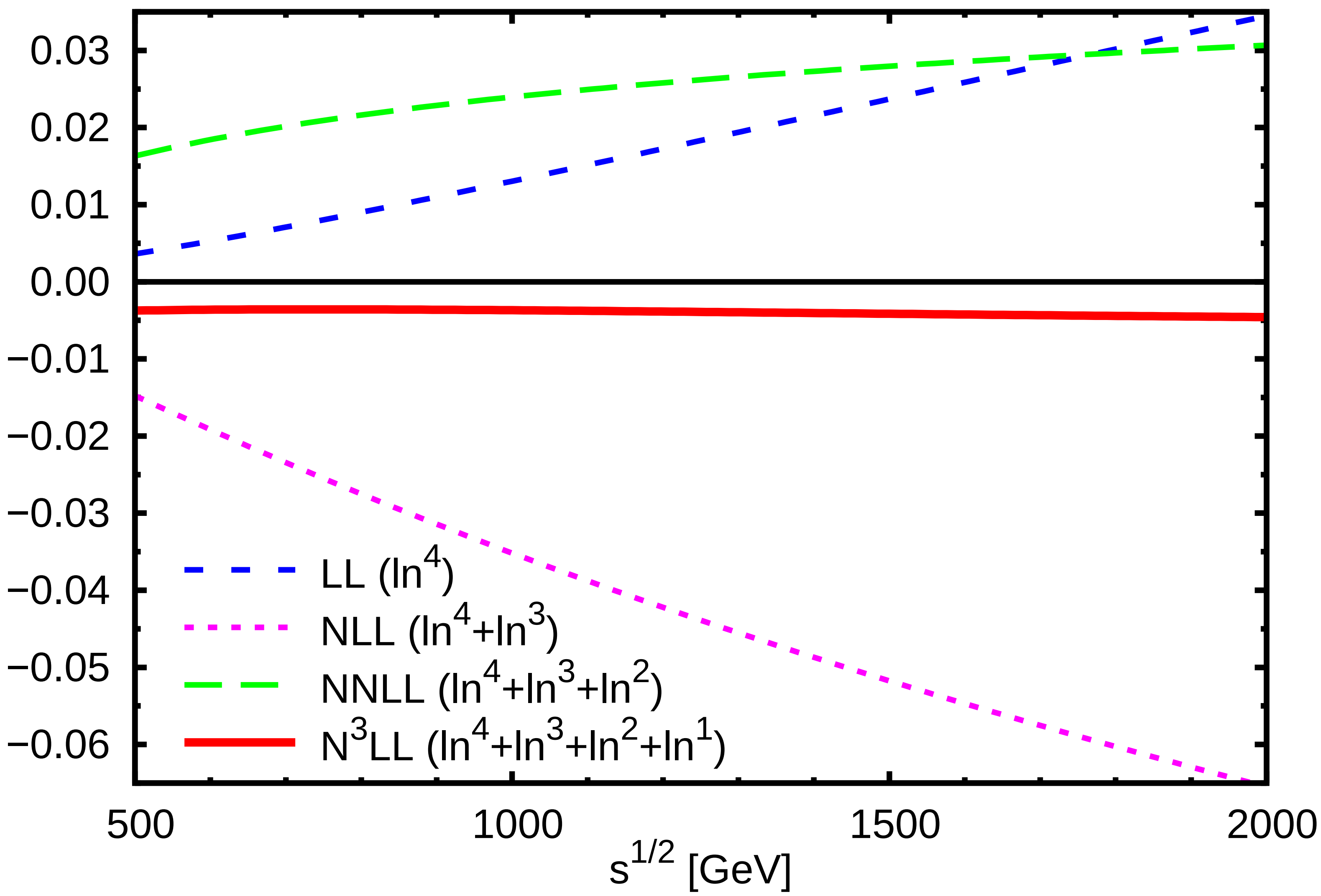
SU(2) form factor in two loops: result (2)



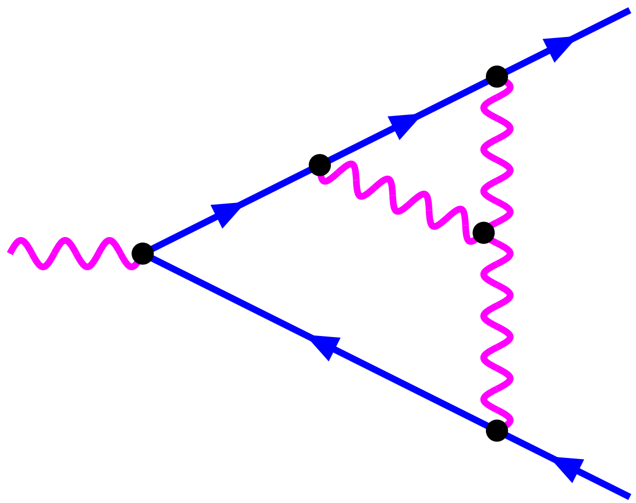
Electroweak results: example $\sigma(e^+e^- \to q\bar{q})$ $(q = d, s)$

numerical 2-loop result:

$$\left(\frac{\alpha_{\text{ew}}}{4\pi}\right)^2 \left[+2.79 \ln^4\left(\frac{s}{M_W^2}\right) - 51.98 \ln^3\left(\frac{s}{M_W^2}\right) + 321.34 \ln^2\left(\frac{s}{M_W^2}\right) - 757.35 \ln\left(\frac{s}{M_W^2}\right) \right]$$



III Evaluating Feynman diagrams in the high energy limit



$$= \, ? + \mathcal{O}\left(\frac{M^2}{Q^2}\right)$$

Reduction to scalar diagrams

- **given** from Feynman rules: $\mathcal{F}^\mu = \bar{u}(p_2)\,\Gamma^\mu(p_1,p_2)\,u(p_1)$
- **wanted:** form factor F with $\mathcal{F}^\mu = F \cdot \bar{u}(p_2)\,\gamma^\mu\,u(p_1)$
- can be calculated using the properties of Dirac matrices and spinors, $\{\gamma^\mu,\gamma^\nu\} = 2g^{\mu\nu}$, $\not{p}_1\,u(p_1) = 0$, $\bar{u}(p_2)\,\not{p}_2 = 0$, combined with tensor reduction
- more elegantly with a *projector* on the form factor:

$$F = \frac{\mathrm{Tr}\,[\gamma_\mu\,\not{p}_2\,\Gamma^\mu(p_1,p_2)\,\not{p}_1]}{2(d-2)\,q^2}$$

- **output:** form factor F in terms of *scalar Feynman integrals*

$$\int \mathrm{d}^d k_1 \int \mathrm{d}^d k_2 \,\frac{\prod_{j=1}^{N} (\ell_j \cdot \ell'_j)^{\nu_j}}{\prod_{i=1}^{L} ({k'_i}^2 - M_i^2)^{n_i}}$$

with propagators and irreducible scalar products in the numerator

Expansion by regions

Beneke, Smirnov '98

a powerful method for the asymptotic expansion of Feynman diagrams

- **given:** scalar Feynman integral & limit like $Q^2 \gg M^2$
- **wanted:** expansion of the *integral* in M^2/Q^2
- **problem:** direct expansion of the *integrand* leads to (new) IR/UV singularities

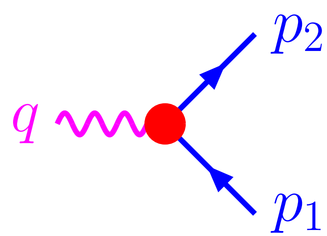
Recipe for the method of expansion by regions:

1. *divide* the integration domain into *regions* for the loop momenta
   (especially such regions where singularities are produced in the limit $M \to 0$)
2. in every region, *expand* the integrand in a *Taylor series* with respect to the parameters that are considered small *there*
3. *integrate* the expanded integrands over the *whole integration domain*
4. put to zero any *integral without scale* (like with dimensional regularization)

- usually only a few regions give non-vanishing contributions
- for logarithmic approximation: only leading order of the expansion needed
  $\hookrightarrow$ in step 2. all small parameters in the integrand are simply set to zero
- sometimes additional regularization (apart from ε) needed for individual regions

Expansion by regions: example

Vertex form factor in the Sudakov limit $Q^2 \gg M^2$



- typical regions for each loop momentum k:

hard	(h):	all components of $k \sim Q$
soft	(s):	all components of $k \sim M$
ultrasoft	(us):	all components of $k \sim M^2/Q$
1-collinear	(1c):	$k^2 \sim 2p_1 \cdot k \sim M^2, \quad 2p_2 \cdot k \sim Q^2$
2-collinear	(2c):	$k^2 \sim 2p_2 \cdot k \sim M^2, \quad 2p_1 \cdot k \sim Q^2$

- 1-loop vertex correction: $f = \dfrac{e^{\varepsilon\gamma_E}}{i\pi^{d/2}} \displaystyle\int \frac{\mathrm{d}^d k}{(k^2 - M^2)(k^2 - 2p_1 \cdot k)(k^2 - 2p_2 \cdot k)}$

$$f^{(h)} = \frac{1}{Q^2} \left[-\frac{1}{\varepsilon^2} + \frac{1}{\varepsilon} \ln(Q^2) - \frac{1}{2} \ln^2(Q^2) + \frac{\pi^2}{12} + \mathcal{O}\left(\frac{M^2}{Q^2}\right) \right]$$

$$f^{(1c)} + f^{(2c)} = \frac{1}{Q^2} \left[\frac{1}{\varepsilon^2} - \frac{1}{\varepsilon} \ln(Q^2) - \frac{1}{2} \ln^2(M^2) + \ln(M^2) \ln(Q^2) - \frac{5}{12}\pi^2 + \mathcal{O}\left(\frac{M^2}{Q^2}\right) \right]$$

$$\Rightarrow f = f^{(h)} + f^{(1c)} + f^{(2c)} = \frac{1}{Q^2} \left[-\frac{1}{2} \ln^2\left(\frac{Q^2}{M^2}\right) - \frac{\pi^2}{3} + \mathcal{O}\left(\frac{M^2}{Q^2}\right) \right]$$

Expansion by regions: how its works

simple 1-dimensional example: $f = \displaystyle\int_0^\infty \frac{\mathrm{d}k\, k^{-\varepsilon}}{(k+m)(k+q)}\,, \quad m \ll q$

$$\left.\begin{array}{lll} \text{soft (s):} & k \sim m, & k < \Lambda \\ \text{hard (h):} & k \sim q, & k > \Lambda \end{array}\right\} \text{where } m \ll \Lambda \ll q$$

$$
\begin{aligned}
f &= \int_0^\Lambda \frac{\mathrm{d}k\, k^{-\varepsilon}}{(k+m)(k+q)} + \int_\Lambda^\infty \frac{\mathrm{d}k\, k^{-\varepsilon}}{(k+m)(k+q)} \\
&= \sum_{j=0}^\infty \frac{(-1)^j}{q^{j+1}} \int_0^\Lambda \frac{\mathrm{d}k\, k^{-\varepsilon+j}}{k+m} + \sum_{i=0}^\infty (-m)^i \int_\Lambda^\infty \frac{\mathrm{d}k\, k^{-\varepsilon-i-1}}{k+q} \\
&= \sum_{j=0}^\infty \frac{(-1)^j}{q^{j+1}} \left(\int_0^\infty \frac{\mathrm{d}k\, k^{-\varepsilon+j}}{k+m} - \int_\Lambda^\infty \frac{\mathrm{d}k\, k^{-\varepsilon+j}}{k+m} \right) + \sum_{i=0}^\infty (-m)^i \left(\int_0^\infty \frac{\mathrm{d}k\, k^{-\varepsilon-i-1}}{k+q} - \int_0^\Lambda \frac{\mathrm{d}k\, k^{-\varepsilon-i-1}}{k+q} \right) \\
&= \underbrace{\sum_{j=0}^\infty \frac{(-1)^j}{q^{j+1}} \int_0^\infty \frac{\mathrm{d}k\, k^{-\varepsilon+j}}{k+m}}_{f^{(s)}} + \underbrace{\sum_{i=0}^\infty (-m)^i \int_0^\infty \frac{\mathrm{d}k\, k^{-\varepsilon-i-1}}{k+q}}_{f^{(h)}} - \sum_{i=0}^\infty (-m)^i \sum_{j=0}^\infty \frac{(-1)^j}{q^{j+1}} \underbrace{\int_0^\infty \mathrm{d}k\, k^{-\varepsilon-i+j-1}}_{\to 0,\ \text{scaleless integral}} \\
&= f^{(s)} + f^{(h)} = \frac{\Gamma(\varepsilon)\Gamma(1-\varepsilon)}{m^\varepsilon\, q} \sum_{j=0}^\infty \left(\frac{m}{q}\right)^j + \frac{\Gamma(-\varepsilon)\Gamma(1+\varepsilon)}{q^{1+\varepsilon}} \sum_{i=0}^\infty \left(\frac{m}{q}\right)^i \\
&= \frac{\Gamma(\varepsilon)\Gamma(1-\varepsilon)}{(q-m)\, m^\varepsilon} + \frac{\Gamma(-\varepsilon)\Gamma(1+\varepsilon)}{(q-m)\, q^\varepsilon} = \frac{\ln(q/m)}{q-m} + \mathcal{O}(\varepsilon) \quad \checkmark
\end{aligned}
$$

Parameterization of Feynman integrals

- Schwinger parameters:

$$\frac{1}{A^n} = \frac{1}{i^n\,\Gamma(n)} \int_0^\infty \mathrm{d}\alpha\,\alpha^{n-1}\,e^{i\alpha A}\,, \quad \text{numerator } A^n = \left(\frac{1}{i}\frac{\partial}{\partial\alpha}\right)^n e^{i\alpha A}\bigg|_{\alpha=0}$$

$\Rightarrow$ any number of propagators and numerators may be combined

$\Rightarrow$ can always be transformed to (generalized) Feynman parameters

$\hookrightarrow$ evaluation:

$$\int \mathrm{d}^d k\, e^{i(\alpha k^2 + 2p\cdot k)} = i\pi^{d/2}\,(i\alpha)^{-d/2}\,e^{-ip^2/\alpha}$$

$$\int_0^\infty \frac{\mathrm{d}\alpha\,\alpha^{n-1}}{(A+\alpha B)^r} = \frac{\Gamma(n)\,\Gamma(r-n)}{\Gamma(r)\,A^{r-n}\,B^n}$$

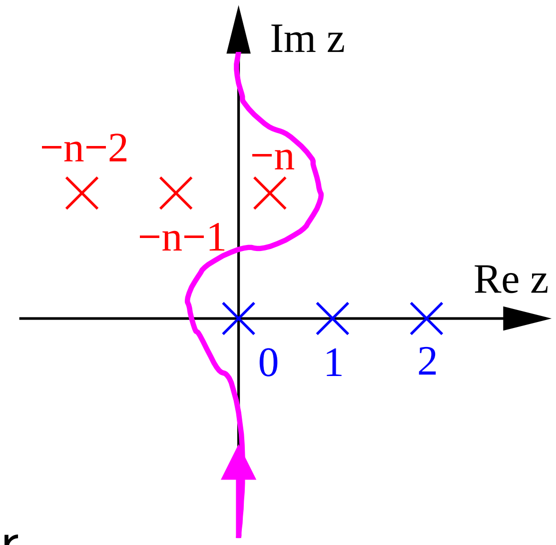
- generalized Feynman parameters:

$$\prod_{i=1}^{L} \frac{1}{A_i^{n_i}} = \frac{\Gamma(\sum_i n_i)}{\prod_i \Gamma(n_i)} \left(\prod_i \int_0^\infty \mathrm{d}x_i\, x_i^{n_i-1}\right) \frac{\delta\Big(\sum_{j\in S} x_j - 1\Big)}{(\sum_i x_i A_i)^{\sum_i n_i}}\,, \qquad \emptyset \neq S \subseteq \{1,\dots,L\}$$

$\Rightarrow$ convenient also for non-standard propagators, e.g. $A_i = 2p\cdot k$

Mellin–Barnes representation

Ussyukina '75; Boos, Davydychev '91



Feynman integrals with many parameters are hard to evaluate

$\hookrightarrow$ separate parameters by Mellin–Barnes representation:

$$\frac{1}{(A+B)^n} = \frac{1}{\Gamma(n)} \int_{-i\infty}^{i\infty} \frac{\mathrm{d}z}{2\pi i} \, \Gamma(-z) \, \Gamma(n+z) \, \frac{B^z}{A^{n+z}}$$

- Mellin–Barnes integrals go along the imaginary axis, leaving poles of $\Gamma(-z+\ldots)$ to the right and poles of $\Gamma(z+\ldots)$ to the left of the integration contour
- applicable to massive propagators ($A = k^2$, $B = -M^2$) or to any complicated parameter integral
- evaluation: close the integration contour to the right ($|B| \leq |A|$) or to the left ($|B| \geq |A|$) and pick up the residues within the contour: $\mathrm{Res}\,\Gamma(z)\big|_{z=-i} = (-1)^i/i!$
- applicable for asymptotic expansions: sum of residues yields expansion in powers of (B/A) or (A/B) and $\ln(A/B)$
- close link to *expansion by regions*: Mellin–Barnes representation of the full integral $\longrightarrow$ sum of residues $\hookrightarrow$ asymptotic expansion with contributions corresponding to the regions

Mellin–Barnes integrals: extraction of singularities

$$I = \int_{-i\infty}^{i\infty} \frac{\mathrm{d}z}{2\pi i} \underbrace{\Gamma(\alpha - z)}_{\text{right poles}} \underbrace{\Gamma(-\alpha + \varepsilon + z)}_{\text{left poles}} f(z)$$

$\Rightarrow$ The right pole at $z = \alpha$ and the left pole at $z = \alpha - \varepsilon$ "glue together" for $\varepsilon \to 0$.
Close contour to the right:

$$-\mathrm{Res}\, \Gamma(\alpha - z)\Gamma(-\alpha + \varepsilon + z) f(z)\Big|_{z=\alpha} = \Gamma(\varepsilon) f(\alpha) = \frac{1}{\varepsilon} f(\alpha) + \mathcal{O}(\varepsilon^0)$$

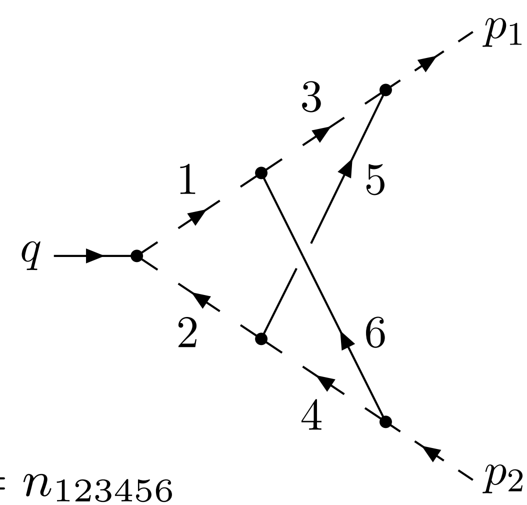
$\Rightarrow$ When a left pole and a right pole glue together, a singularity is produced!

Extraction of such singularities:

$$I = -\mathrm{Res}\, \Gamma(\alpha - z)\Gamma(-\alpha+\varepsilon+z) f(z)\Big|_{z=\alpha} + \int_{-i\infty}^{i\infty} \frac{\mathrm{d}z}{2\pi i} \underbrace{\Gamma(1 + \alpha - z)}_{\text{right poles}} \underbrace{\frac{\Gamma(-\alpha + \varepsilon + z)}{\alpha - z}}_{\text{left poles}} f(z)$$

Now the poles at $z = \alpha$ and $z = \alpha - \varepsilon$ both lie to the left of the integration contour.
$\hookrightarrow$ The integrand can safely be expanded in ε.

Example: the non-planar vertex diagram



Scalar integrals with variable powers of propagators:

$$F_{\mathsf{NP}}(n_1,\ldots,n_7) = e^{2\varepsilon\gamma}\,(M^2)^{2\varepsilon}\,(Q^2)^{n-n_7-4}$$

$$\times \int \frac{\mathrm{d}^d k}{i\pi^{d/2}} \int \frac{\mathrm{d}^d \ell}{i\pi^{d/2}}\, \frac{(2k\cdot\ell)^{n_7}}{\big((p_1-k-\ell)^2\big)^{n_1}\,\big((p_2-k-\ell)^2\big)^{n_2}}$$

$$\times \frac{1}{(k^2-2p_1\cdot k)^{n_3}\,(\ell^2-2p_2\cdot\ell)^{n_4}\,(k^2-M^2)^{n_5}\,(\ell^2-M^2)^{n_6}}\,, \quad n = n_{123456}$$

Contributing regions: (h-h), (1c-h), (h-2c), (1c-1c), (2c-2c), (1c-2c), (1c-1c'), (2c'-2c), (us'-us'), (1c-us'), (us'-2c).

Leading term of (1c-h) region $\iff$ $k^2 \sim 2p_1\cdot k \sim M^2$, $2p_2\cdot k \sim Q^2$, $\ell \sim Q$:

$$F_{\mathsf{NP}}^{(\mathsf{1c\text{-}h})}(n_1,\ldots,n_7) = e^{2\varepsilon\gamma}\,(M^2)^{2\varepsilon}\,(Q^2)^{n-n_7-4} \int \frac{\mathrm{d}^d k}{i\pi^{d/2}} \int \frac{\mathrm{d}^d \ell}{i\pi^{d/2}}$$

$$\times \frac{\big((2p_2\cdot k)(2p_1\cdot\ell)/Q^2\big)^{n_7}}{\big(\ell^2-2p_1\cdot\ell+(2p_2\cdot k)(2p_1\cdot\ell)/Q^2\big)^{n_1}\,\big(\ell^2-2p_2\cdot(k+\ell)+(2p_2\cdot k)(2p_1\cdot\ell)/Q^2\big)^{n_2}}$$

$$\times \frac{1}{(k^2-2p_1\cdot k)^{n_3}\,(\ell^2-2p_2\cdot\ell)^{n_4}\,(k^2-M^2)^{n_5}\,(\ell^2)^{n_6}} + \mathcal{O}\bigg(\frac{M^2}{Q^2}\bigg)$$

Example: (1c-h) region of the non-planar vertex diagram

Introduce Feynman or Schwinger parameters, integrate & transform into

$$F_{\mathrm{NP}}^{(\text{1c-h})}(n_1,\ldots,n_7) = \left(\frac{M^2}{Q^2}\right)^{2-n_{35}+\varepsilon} (-1)^n \, e^{2\varepsilon\gamma} \, \frac{\Gamma(\frac{d}{2}-n_{24})\Gamma(\frac{d}{2}-n_{16}+n_7)\Gamma(n_{35}-\frac{d}{2})}{\Gamma(d-n_{1246}+n_7)\prod_{i=1}^{6}\Gamma(n_i)}$$

$$\times \int_0^1 \mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3 \, x_1^{n_1-1}(1-x_1)^{n_6-1}x_2^{n_2-1}(1-x_2)^{n_4-1}x_3^{n_{37}-1}(1-x_3)^{\frac{d}{2}-n_3-1}$$

$$\times \underbrace{\Gamma(n_{1246}-\tfrac{d}{2})\left[x_1(1-x_3)+x_2x_3\right]^{\frac{d}{2}-n_{1246}}}_{\text{Mellin–Barnes representation:}}$$

$$\int_{-i\infty}^{i\infty}\frac{\mathrm{d}z}{2\pi i}\,\Gamma(-z)\Gamma(n_{1246}-\tfrac{d}{2}+z)\left(x_1(1-x_3)\right)^z (x_2x_3)^{\frac{d}{2}-n_{1246}-z}$$

$\Rightarrow$ Expression with Mellin–Barnes integral:

$$F_{\mathrm{NP}}^{(\text{1c-h})}(n_1,\ldots,n_7) = \left(\frac{M^2}{Q^2}\right)^{2-n_{35}+\varepsilon} (-1)^n \, \frac{e^{2\varepsilon\gamma}\,\Gamma(\frac{d}{2}-n_{24})\Gamma(\frac{d}{2}-n_{16}+n_7)\Gamma(n_{35}-\frac{d}{2})}{\Gamma(n_1)\Gamma(n_2)\Gamma(n_3)\Gamma(n_5)\,\Gamma(d-n_{1246}+n_7)^2}$$

$$\times \int_{-i\infty}^{i\infty}\frac{\mathrm{d}z}{2\pi i}\,\frac{\Gamma(-z)\Gamma(\frac{d}{2}-n_{146}-z)\Gamma(\frac{d}{2}-n_{1246}+n_{37}-z)}{\Gamma(\frac{d}{2}-n_{16}-z)}$$

$$\times \frac{\Gamma(n_1+z)\Gamma(\frac{d}{2}-n_3+z)\Gamma(n_{1246}-\frac{d}{2}+z)}{\Gamma(n_{16}+z)}$$

Example: evaluation of the (1c-h) region for special cases (1)

$$F_{\mathsf{NP}}^{(\mathsf{1c\text{-}h})}(1,1,1,0,1,0,0) = \left(\frac{M^2}{Q^2}\right)^{\varepsilon} e^{2\varepsilon\gamma}\, \frac{\Gamma(1-\varepsilon)^2\Gamma(\varepsilon)}{\Gamma(2-2\varepsilon)^2} \times \int_{-i\infty}^{i\infty} \frac{\mathrm{d}z}{2\pi i}\, \Gamma(-z)\Gamma(1-\varepsilon-z)\,\Gamma(1-\varepsilon+z)\Gamma(\varepsilon+z)$$

Solution known: 1st Barnes lemma — Barnes 1908

$$\int_{-i\infty}^{i\infty} \frac{\mathrm{d}z}{2\pi i}\, \Gamma(\alpha_1-z)\Gamma(\alpha_2-z)\,\Gamma(\alpha_3+z)\Gamma(\alpha_4+z) = \frac{\Gamma(\alpha_1+\alpha_3)\Gamma(\alpha_1+\alpha_4)\Gamma(\alpha_2+\alpha_3)\Gamma(\alpha_2+\alpha_4)}{\Gamma(\alpha_1+\alpha_2+\alpha_3+\alpha_4)}$$

$$\Rightarrow F_{\mathsf{NP}}^{(\mathsf{1c\text{-}h})}(1,1,1,0,1,0,0) = \left(\frac{M^2}{Q^2}\right)^{\varepsilon} e^{2\varepsilon\gamma}\, \frac{\Gamma(1-\varepsilon)^2\Gamma(\varepsilon)}{\Gamma(2-2\varepsilon)^2}\, \frac{\Gamma(1-\varepsilon)\Gamma(\varepsilon)\Gamma(2-2\varepsilon)}{\Gamma(2-\varepsilon)}$$

$$= \frac{1}{\varepsilon^2} + \frac{1}{\varepsilon}\Big(-\mathcal{L}+3\Big) + \frac{1}{2}\mathcal{L}^2 - 3\mathcal{L} + 7 + \mathcal{O}(\varepsilon)\,, \quad \mathcal{L} = \ln\!\left(\frac{Q^2}{M^2}\right)$$

Example: evaluation of the (1c-h) region for special cases (2)

$$F_{\mathsf{NP}}^{(\mathsf{1c\text{-}h})}(1,1,1,0,1,1,0) = -\left(\frac{M^2}{Q^2}\right)^{\varepsilon} e^{2\varepsilon\gamma}\,\frac{\Gamma(1-\varepsilon)\Gamma(-\varepsilon)\Gamma(\varepsilon)}{\Gamma(1-2\varepsilon)^2}$$
$$\times \int_{-i\infty}^{i\infty}\frac{\mathrm{d}z}{2\pi i}\,\Gamma(-z)\Gamma(-\varepsilon-z)\,\frac{\Gamma(1+z)\Gamma(1-\varepsilon+z)\Gamma(1+\varepsilon+z)}{\Gamma(2+z)}$$

- 1st possibility: expand integrand in ε
  $\hookrightarrow$ transform to expressions solvable by Barnes lemma etc.
- 2nd possibility: close integration contour to the right and take residues directly:

$$\left(\frac{M^2}{Q^2}\right)^{\varepsilon} e^{2\varepsilon\gamma}\,\frac{\Gamma(1-\varepsilon)^3\Gamma(1+\varepsilon)^2}{\varepsilon^3\,\Gamma(1-2\varepsilon)^2}\sum_{i=0}^{\infty}\Bigg[\underbrace{\frac{\Gamma(1-2\varepsilon+i)}{\Gamma(2-\varepsilon+i)}}_{\text{from } z=-\varepsilon+i} - \underbrace{\frac{\Gamma(1-\varepsilon+i)}{\Gamma(2+i)}}_{\text{from } z=i}\Bigg]$$

expand Gamma functions $\rightarrow$ sum up to (multiple) zeta values:

$$F_{\mathsf{NP}}^{(\mathsf{1c\text{-}h})}(1,1,1,0,1,1,0) = \frac{\pi^2}{6\varepsilon^2} + \frac{1}{\varepsilon}\left(-\frac{\pi^2}{6}\mathcal{L} + 2\zeta_3\right) + \frac{\pi^2}{12}\mathcal{L}^2 - 2\zeta_3\mathcal{L} + \frac{\pi^4}{40} + \mathcal{O}(\varepsilon)$$

Example: evaluation of the (1c-h) region for special cases (3)

$$F_{\mathsf{NP}}^{\text{(1c-h)}}(1,\delta,1,1,1,1,0) = -\left(\frac{M^2}{Q^2}\right)^{\varepsilon} (-1)^{\delta}\, e^{2\varepsilon\gamma}\, \frac{\Gamma(1-\varepsilon-\delta)\Gamma(-\varepsilon)\Gamma(\varepsilon)}{\Gamma(\delta)\Gamma(1-2\varepsilon-\delta)^2}$$

$$\times \int_{-i\infty}^{i\infty} \frac{\mathrm{d}z}{2\pi i}\, \frac{\Gamma(-z)\Gamma(-1-\varepsilon-z)\Gamma(-\varepsilon-\delta-z)}{\Gamma(-\varepsilon-z)}\, \frac{\Gamma(1+z)\Gamma(1-\varepsilon+z)\Gamma(1+\varepsilon+\delta+z)}{\Gamma(2+z)}$$

Limit $\delta \to 0 \Rightarrow \dfrac{1}{\Gamma(\delta)} \to 0$, but gluing poles at $z=-1-\varepsilon$ and $z=-1-\varepsilon-\delta$:

$$F_{\mathsf{NP}}^{\text{(1c-h)}}(1,0,1,1,1,1,0) = \lim_{\delta\to 0}\left(\frac{M^2}{Q^2}\right)^{\varepsilon} (-1)^{\delta}\, e^{2\varepsilon\gamma}\, \frac{\Gamma(1-\varepsilon-\delta)\Gamma(-\varepsilon)\Gamma(\varepsilon)}{\Gamma(\delta)\Gamma(1-2\varepsilon-\delta)^2}$$

$$\times \operatorname{Res} \frac{\Gamma(-z)\Gamma(-1-\varepsilon-z)\Gamma(-\varepsilon-\delta-z)}{\Gamma(-\varepsilon-z)}\, \frac{\Gamma(1+z)\Gamma(1-\varepsilon+z)\Gamma(1+\varepsilon+\delta+z)}{\Gamma(2+z)}\Bigg|_{z=-1-\varepsilon}$$

$$= -\left(\frac{M^2}{Q^2}\right)^{\varepsilon} e^{2\varepsilon\gamma}\, \frac{\Gamma(1-\varepsilon)\Gamma(-\varepsilon)\Gamma(\varepsilon)}{\Gamma(1-2\varepsilon)^2}\, \frac{\Gamma(1+\varepsilon)\Gamma(-\varepsilon)\Gamma(-2\varepsilon)}{\Gamma(1-\varepsilon)}$$

$$= \frac{1}{2\varepsilon^4} - \frac{1}{2\varepsilon^3}\mathcal{L} + \frac{1}{4\varepsilon^2}\mathcal{L}^2 - \frac{1}{\varepsilon}\left(\frac{1}{12}\mathcal{L}^3 + \frac{4}{3}\zeta_3\right) + \frac{1}{48}\mathcal{L}^4 + \frac{4}{3}\zeta_3\mathcal{L} - \frac{\pi^4}{60} + \mathcal{O}(\varepsilon)$$

IV Summary & outlook

Four-fermion scattering

- calculation in the high energy limit reduced to the SU(2) form factor
- 2-loop form factor calculated including all large logarithms
- electroweak 2-loop corrections obtained in N^{3}LL approximation

Evaluating Feynman diagrams in the high energy limit

- expansion by regions
- Mellin–Barnes representation

$\hookrightarrow$ effective combination of advanced methods

Outlook: more legs & more scales

apply methods for Feynman diagrams with

- more than 3 external legs $\rightarrow$ depend on all $p_i \cdot p_j$
- different heavy masses M_W, M_Z, M_{Higgs}, m_t, $\ldots$

$\Rightarrow$ successful extraction of LL and NLL contribution from Mellin–Barnes integrals ✓