

Seminar über spezielle Fragen der theoretischen Elementarteilchenphysik

RWTH Aachen University

February 5, 2009

Two-loop electroweak next-to-leading logarithms for processes involving heavy quarks

Bernd Jantzen

RWTH Aachen University

In collaboration with **Ansgar Denner** and **Stefano Pozzorini**

massless fermions: Nucl. Phys. B761 (2007) 1–62 [arXiv:hep-ph/0608326]

heavy quarks: JHEP 11 (2008) 062 [arXiv:0809.0800 [hep-ph]]

Overview

I Electroweak (EW) corrections at high energies

- origin and importance of EW logs
- existing virtual EW 2-loop corrections

II Two-loop next-to-leading logarithmic (NLL) corrections

- extraction of NLL logs
- evaluation of Feynman diagrams:
expansion by regions & Mellin–Barnes representation
- new features with massive fermions

III Results for processes involving massless and massive fermions

- factorization and exponentiation of the logs
- comparisons and applications
- structure of the result to all orders in ϵ

IV Summary & outlook

Overview

I Electroweak (EW) corrections at high energies

- origin and importance of EW logs
- existing virtual EW 2-loop corrections

II Two-loop next-to-leading logarithmic (NLL) corrections

- extraction of NLL logs
- evaluation of Feynman diagrams:
expansion by regions & Mellin–Barnes representation
- new features with massive fermions

III Results for processes involving massless and massive fermions

- factorization and exponentiation of the logs
- comparisons and applications
- structure of the result to all orders in ϵ

IV Summary & outlook

I Electroweak corrections at high energies

Precision collider physics

Precision measurements at colliders and theoretical predictions with electroweak (EW) + QCD corrections enable us to test the Standard Model at various energy scales:

- up to now (LEP, Tevatron) at energy scales $\lesssim M_{W,Z}$
- future colliders (LHC, ILC/CLIC) \rightarrow reach TeV regime
 \hookrightarrow new energy domain $\sqrt{s} \gg M_W$ becomes accessible!

EW radiative corrections at high energies $\sqrt{s} \gg M_W$

\Rightarrow enhanced by large Sudakov logarithms

$$\text{per loop: } \ln^2 \left(\frac{s}{M_W^2} \right) \sim 25 \quad \text{at } \sqrt{s} \sim 1 \text{ TeV}$$

\hookrightarrow corrections rise with energy

Origin of large EW logs

- **mass singularities**: real or virtual emission of **soft/collinear gauge bosons** from external particles
- remnants from **UV singularities**: $\ln(s/\mu_R^2)$

Massless gauge bosons

real emission of soft/collinear photons/gluons **cannot be detected separately**

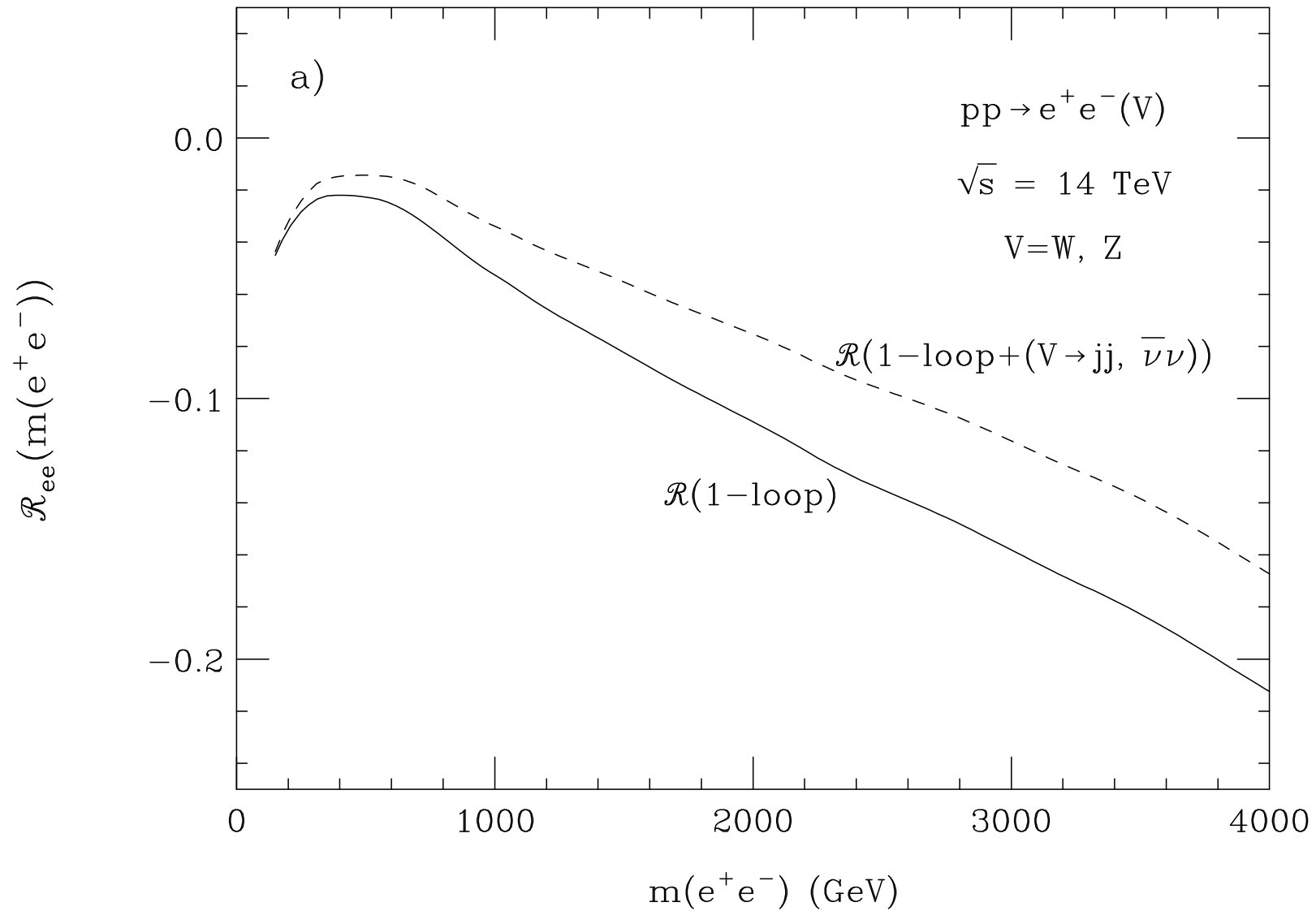
↪ mass singularities cancel between real & virtual corrections (**KLN theorem**)

Massive gauge bosons

real emission of W 's, Z 's **can be detected separately**

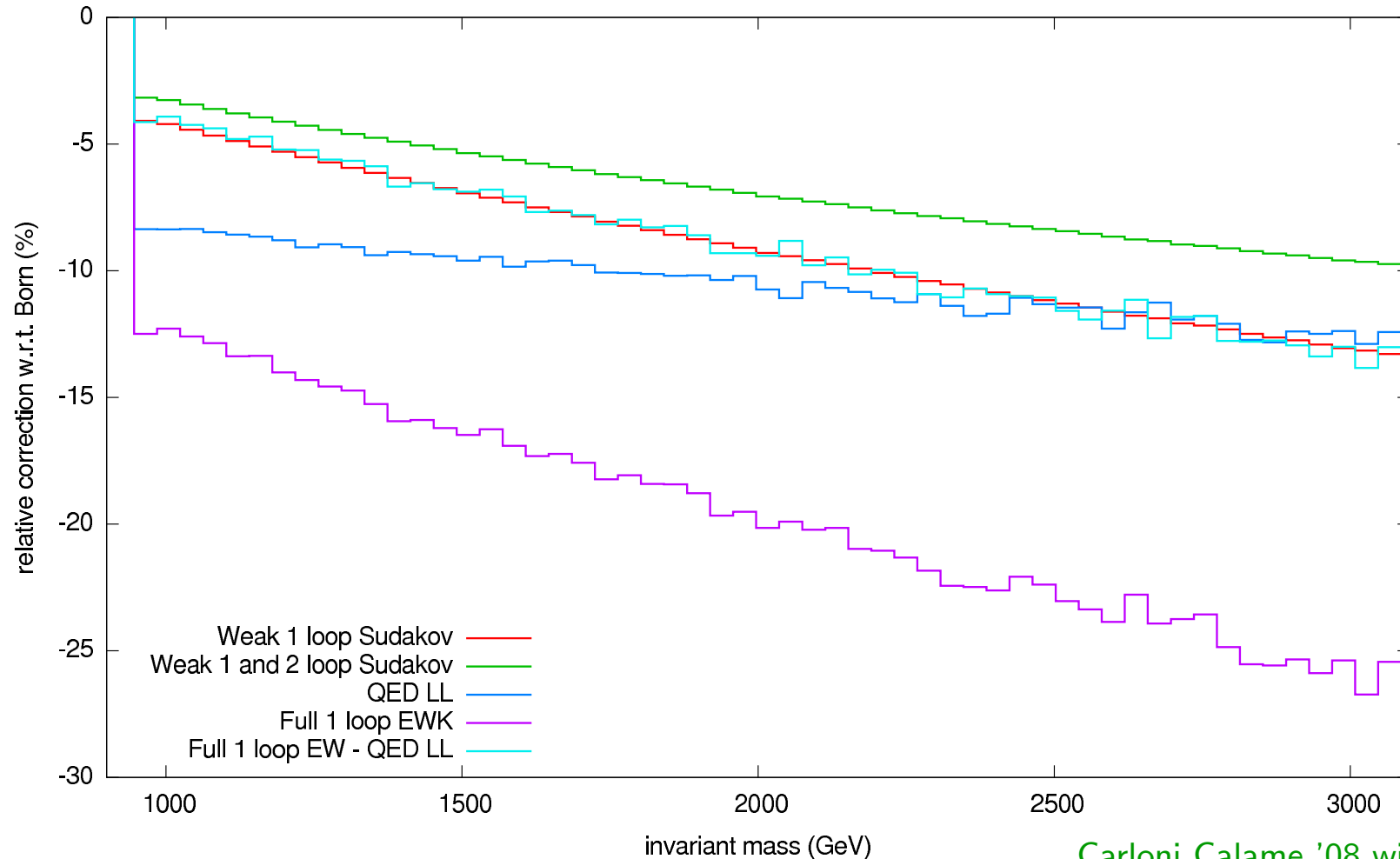
↪ large logs remain present in **exclusive observables** (only virtual corrections),

↪ even in inclusive observables (**Bloch–Nordsieck violations**)

Virtual + Real W, Z emission: only partial cancellation

EW 1-/2-loop corrections at the LHC

Drell–Yan $pp \rightarrow \mu^+ \mu^-$: (electro)weak 1-loop & 2-loop corrections



Carlani Calame '08 with HORACE
and logarithmic ("Sudakov") results from
B.J., Kühn, Penin, Smirnov '05

⇒ logarithmic approximation very good at high energies

⇒ 2-loop effects $\sim \mathcal{O}(\%)$

cf. Les Houches 2007 report, arXiv:0803.0678 [hep-ph]

General form of virtual EW corrections for $s \gg M_W^2$ $\left[L = \ln \left(\frac{s}{M_W^2} \right) \right]$

\hookrightarrow LL (leading logarithmic), NLL (next-to-leading logarithmic), N²LL ... terms:

$$\mathbf{1 \text{ loop:}} \quad \alpha \left[C_1^{\text{LL}} L^2 + C_1^{\text{NLL}} L + C_1^{\text{N}^2\text{LL}} \right] + \mathcal{O} \left(\frac{M_W^2}{s} \right)$$

$$\begin{array}{ccc} \downarrow & \downarrow & \downarrow \\ -17\% & +12\% & -3\% \end{array}$$

$$\mathbf{2 \text{ loops:}} \quad \alpha^2 \left[C_2^{\text{LL}} L^4 + C_2^{\text{NLL}} L^3 + C_2^{\text{N}^2\text{LL}} L^2 + C_2^{\text{N}^3\text{LL}} L + C_2^{\text{N}^4\text{LL}} \right] + \mathcal{O} \left(\frac{M_W^2}{s} \right)$$

$$\begin{array}{cccc} \downarrow & \downarrow & \downarrow & \downarrow \\ +1.7\% & -1.8\% & +1.2\% & -0.3\% \end{array}$$

$[\sigma(u\bar{u} \rightarrow d\bar{d}) @ \sqrt{s} = 1 \text{ TeV, B.J., Kühn, Penin, Smirnov '05}]$

\hookrightarrow large cancellations!

For theoretical predictions with accuracy $\sim 1\%$:

\Rightarrow 2-loop corrections important

\Rightarrow LL approximation not sufficient

With massless photons: $\log \rightsquigarrow 1/\epsilon$ in $D = 4 - 2\epsilon$ dimensions

Existing virtual EW 2-loop corrections

Resummation of 1-loop results using **evolution equations** valid for unbroken $SU(2) \times U(1)$ ($M_\gamma = M_Z = M_W$) and for pure QED:

$$\alpha^2 \left[\underbrace{C_{LL} \ln^4 \left(\frac{s}{M_W^2} \right)}_{\substack{\text{Fadin, Lipatov,} \\ \text{Martin, Melles '99}}} + \underbrace{C_{NLL} \ln^3 \left(\frac{s}{M_W^2} \right)}_{\text{Melles '00, '01}} + \underbrace{C_{N^2LL} \ln^2 \left(\frac{s}{M_W^2} \right)}_{\substack{\text{Kühn, Moch, Penin,} \\ \text{Smirnov '99–'01}}} + \underbrace{C_{N^3LL} \ln \left(\frac{s}{M_W^2} \right)}_{\substack{\text{B.J., Kühn, Moch, Penin,} \\ \text{Smirnov '03–'05}}} \right]$$

arbitrary processes
massless $f \bar{f} \rightarrow f' \bar{f}'$ processes

+ SCET approach [Chiu, Golf, Kelley, Manohar '07, '08]

+ N^2LL for $e^+ e^- \rightarrow W^+ W^-$ [Kühn, Metzler, Penin '07]

Explicit 2-loop calculations based on spontaneously broken $SU(2) \times U(1)$, $M_Z \neq M_W$:

$$\alpha^2 \left[\underbrace{C_{LL} \ln^4 \left(\frac{s}{M_W^2} \right)}_{\substack{\text{Melles '00;} \\ \text{Hori, Kawamura, Kodaira '00;} \\ \text{Beenakker, Werthenbach '00, '01}}} + \underbrace{C_{NLL}^{\text{ang}} \ln \left(\frac{t}{s} \right) \ln^3 \left(\frac{s}{M_W^2} \right)}_{\text{Denner, Melles, Pozzorini '03}} + \underbrace{C_{NLL}^{\text{rem}} \ln^3 \left(\frac{s}{M_W^2} \right)}_{\substack{\text{Pozzorini '04;} \\ \text{Denner, B.J., Pozzorini '06, '08}}} \right]$$

arbitrary processes
 n -fermion processes

\Rightarrow agreement with evolution equations \checkmark

Overview

I Electroweak (EW) corrections at high energies

- origin and importance of EW logs
- existing virtual EW 2-loop corrections

II Two-loop next-to-leading logarithmic (NLL) corrections

- extraction of NLL logs
- evaluation of Feynman diagrams:
expansion by regions & Mellin–Barnes representation
- new features with massive fermions

III Results for processes involving massless and massive fermions

- factorization and exponentiation of the logs
- comparisons and applications
- structure of the result to all orders in ϵ

IV Summary & outlook

II Two-loop next-to-leading logarithmic corrections

Goal: virtual EW 2-loop corrections for arbitrary processes in NLL accuracy

↪ start with processes involving massless & massive external fermions

Parameters:

- different large kinematical invariants $r_{ij} = (p_i + p_j)^2 \sim Q^2 \gg M_W^2$
- different heavy particle masses $M_W^2 \sim M_Z^2 \sim m_t^2 \sim M_{\text{Higgs}}^2$
- massive top quark, other fermions massless

⇒ logs $L = \ln \left(\frac{Q^2}{M_W^2} \right)$ and $\frac{1}{\epsilon}$ poles (from virtual photons, counted like logs)

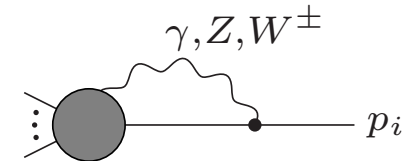
1 loop: LL $\rightarrow \epsilon^{-2}, L\epsilon^{-1}, L^2, L^3\epsilon, L^4\epsilon^2$; NLL $\rightarrow \epsilon^{-1}, L, L^2\epsilon, L^3\epsilon^2$

2 loops: LL $\rightarrow \epsilon^{-4}, L\epsilon^{-3}, L^2\epsilon^{-2}, L^3\epsilon^{-1}, L^4$; NLL $\rightarrow \epsilon^{-3}, L\epsilon^{-2}, L^2\epsilon^{-1}, L^3$

⇒ NLL coefficients involve small logs $\ln \left(\frac{-r_{ij}}{Q^2} \right)$ and $\ln \left(\frac{M_Z^2, m_t^2}{M_W^2} \right)$

Extraction of NLL logs

Logs originate from **mass singularities** when a virtual gauge boson (γ, Z, W^\pm) couples to an **on-shell external leg**
 \rightarrow **single log** from **collinear region** (+ UV logs)

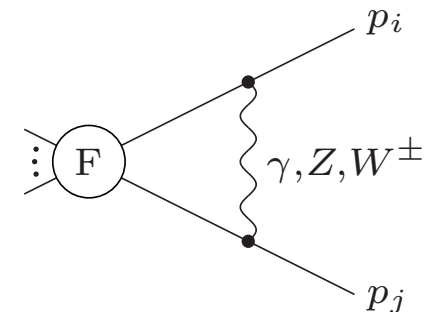


Isolate factorizable contributions:

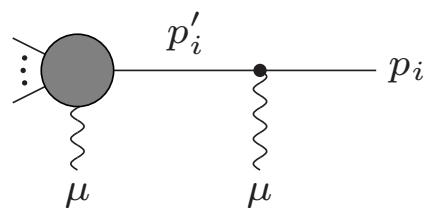
gauge boson exchanged between external legs,

set gauge boson momentum to zero in tree subdiagram $\textcircled{\text{F}}$

\rightarrow **double log** from **soft & collinear region**



Separate loop integral from Born diagram $\textcircled{\text{F}}$ via **soft-collinear approximation:**



$$i\not{p}'_i \cdot i\gamma^\mu \rightarrow -2p'_i{}^\mu \quad (\text{for massless external fermions})$$

\hookrightarrow eliminates Dirac structure of loop corrections

\Rightarrow valid for **soft or collinear** gauge boson momenta

\Rightarrow generalization of *eikonal approximation*

[not applicable for certain diagrams where we had to use projection techniques instead]

Remaining non-factorizable contributions

Contributions from collinear region:

$$\begin{array}{c} \text{---} \\ \vdots \\ \bullet \\ \text{---} \end{array} \text{---} \text{---} \text{---} i \quad - \quad \begin{array}{c} \text{---} \\ \vdots \\ \bullet \\ \text{---} \end{array} \text{---} \text{---} \text{---} i \quad - \quad \sum_{j \neq i} \begin{array}{c} \text{---} \\ \vdots \\ \text{---} \end{array} \text{---} \text{---} \text{---} i \quad = 0$$

Cancellation mechanism:

- collinear vertex \propto gauge boson momentum q^μ
- **collinear Ward identities** for EW theory:

Denner, Pozzorini '00, '01

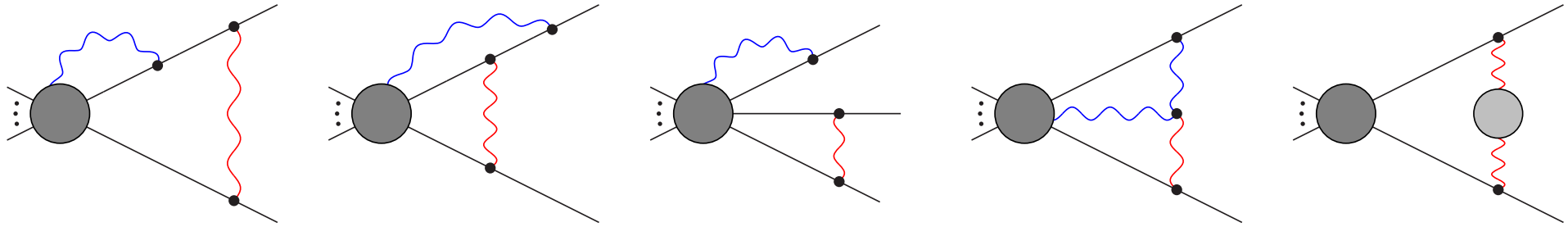
$$q^\mu \times \left\{ \begin{array}{c} \text{---} \\ \vdots \\ \bullet \\ \text{---} \end{array} \text{---} \text{---} \text{---} \mu \quad - \quad \begin{array}{c} \text{---} \\ \vdots \\ \bullet \\ \text{---} \end{array} \text{---} \text{---} \text{---} \mu \quad - \quad \sum_{j \neq i} \begin{array}{c} \text{---} \\ \vdots \\ \text{---} \end{array} \text{---} \text{---} \text{---} \mu \quad \right\} \rightarrow 0$$

in the collinear limit

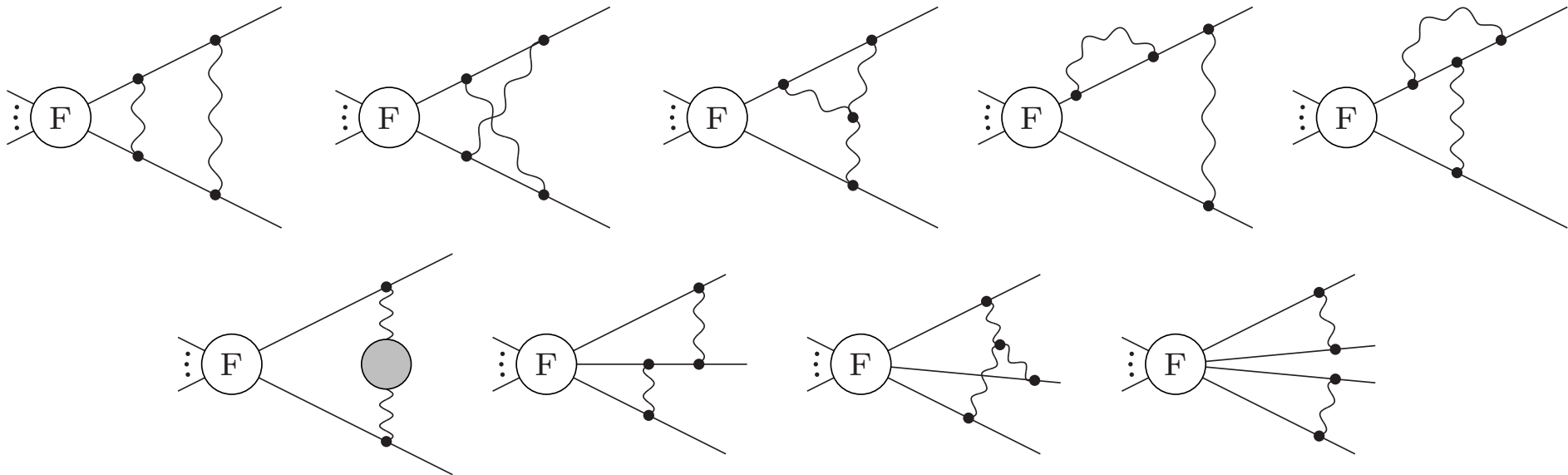
The factorizable contributions contain all soft or collinear NLL mass singularities.

Extraction of NLL logs at 2 loops

$\hookrightarrow L^{4,3}$ contributions: **soft** \times **soft** and **soft** \times **collinear** (without Yukawa contributions)



Factorizable contributions:



- calculated with soft-collinear approximation (and projection techniques)
- non-factorizable contributions vanish

Treatment of ultraviolet (UV) singularities

UV $1/\epsilon$ poles in subdiagrams with scale μ_{loop}^2 & renormalization at scale μ_{R}^2 :

$$\underbrace{\frac{1}{\epsilon} \left(\frac{Q^2}{\mu_{\text{loop}}^2} \right)^\epsilon}_{\text{bare diagrams}} - \underbrace{\frac{1}{\epsilon} \left(\frac{Q^2}{\mu_{\text{R}}^2} \right)^\epsilon}_{\text{counterterms}} = \ln \left(\frac{\mu_{\text{R}}^2}{\mu_{\text{loop}}^2} \right) + \mathcal{O}(\epsilon) \quad \Rightarrow \quad \text{possible NLL contribution}$$

Perform **minimal UV subtraction** in UV-singular (sub)diagrams and counterterms:

$$\underbrace{\frac{1}{\epsilon} \left[\left(\frac{Q^2}{\mu_{\text{loop}}^2} \right)^\epsilon - 1 \right]}_{\text{bare diagrams}} - \underbrace{\frac{1}{\epsilon} \left[\left(\frac{Q^2}{\mu_{\text{R}}^2} \right)^\epsilon - 1 \right]}_{\text{counterterms}}$$

Advantages:

- no UV NLL terms from **hard** subdiagrams ($\mu_{\text{loop}}^2 \sim Q^2$)
 \hookrightarrow no UV contributions from **internal** parts of tree subdiagrams
- can use soft-collinear approximation (not valid in UV regime!)
 also for hard UV-singular subdiagrams

Evaluation of Feynman diagrams

Need to evaluate Feynman diagrams in the **high-energy limit** $Q^2 \gg M^2 \sim M_W^2$

⇒ discuss combination of two methods:

- **expansion by regions,**
- **Mellin–Barnes representations.**

cf. B.J., Smirnov '06 & refs. therein

The same diagrams have also been evaluated independently with an automatized algorithm based on **sector decomposition** (not discussed here).

Denner, Pozzorini '04

Expansion by regions

Beneke, Smirnov '98

a powerful method for the asymptotic expansion of Feynman diagrams

- **given:** (scalar) Feynman integral & limit like $Q^2 \gg M^2$
- **wanted:** expansion of the *integral* in M^2/Q^2
- **problem:** direct expansion of the *integrand* leads to (new) IR/UV singularities

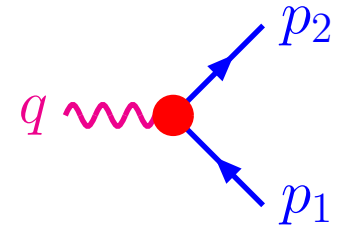
Recipe for the method of expansion by regions:

1. *divide* the integration domain into *regions* for the loop momenta
(especially such regions where singularities are produced in the limit $M \rightarrow 0$)
2. in every region, *expand* the integrand in a *Taylor series* with respect to the parameters that are considered small *there*
3. *integrate* the expanded integrands over the *whole integration domain*
4. put to zero any *integral without scale* (like with dimensional regularization)
 - usually only a few regions give non-vanishing contributions
 - for logarithmic approximation: only leading order of the expansion needed
↔ in step 2. all small parameters in the integrand are simply set to zero
 - sometimes analytic regularization (in addition to ϵ) needed for individual regions

Expansion by regions: example

Vertex form factor in the Sudakov limit $Q^2 \gg M^2$

(massless fermions, gauge boson mass M)



- typical regions for each loop momentum k :

| | | |
|-------------|-------|---|
| hard | (h): | all components of $k \sim Q$ |
| soft | (s): | all components of $k \sim M$ |
| ultrasoft | (us): | all components of $k \sim M^2/Q$ |
| 1-collinear | (1c): | $k^2 \sim 2p_1 \cdot k \sim M^2, \quad 2p_2 \cdot k \sim Q^2$ |
| 2-collinear | (2c): | $k^2 \sim 2p_2 \cdot k \sim M^2, \quad 2p_1 \cdot k \sim Q^2$ |

- 1-loop vertex correction: $f = \frac{e^{\epsilon\gamma_E}}{i\pi^{d/2}} \int \frac{d^d k}{(k^2 - M^2)(k^2 - 2p_1 \cdot k)(k^2 - 2p_2 \cdot k)}$

$$f^{(h)} = \frac{1}{Q^2} \left[-\frac{1}{\epsilon^2} + \frac{1}{\epsilon} \ln(Q^2) - \frac{1}{2} \ln^2(Q^2) + \frac{\pi^2}{12} + \mathcal{O}\left(\frac{M^2}{Q^2}\right) \right]$$

$$f^{(1c)} + f^{(2c)} = \frac{1}{Q^2} \left[\frac{1}{\epsilon^2} - \frac{1}{\epsilon} \ln(Q^2) - \frac{1}{2} \ln^2(M^2) + \ln(M^2) \ln(Q^2) - \frac{5}{12} \pi^2 + \mathcal{O}\left(\frac{M^2}{Q^2}\right) \right]$$

$$\Rightarrow f = f^{(h)} + f^{(1c)} + f^{(2c)} = \frac{1}{Q^2} \left[-\frac{1}{2} \ln^2\left(\frac{Q^2}{M^2}\right) - \frac{\pi^2}{3} + \mathcal{O}\left(\frac{M^2}{Q^2}\right) \right]$$

Mellin–Barnes representation

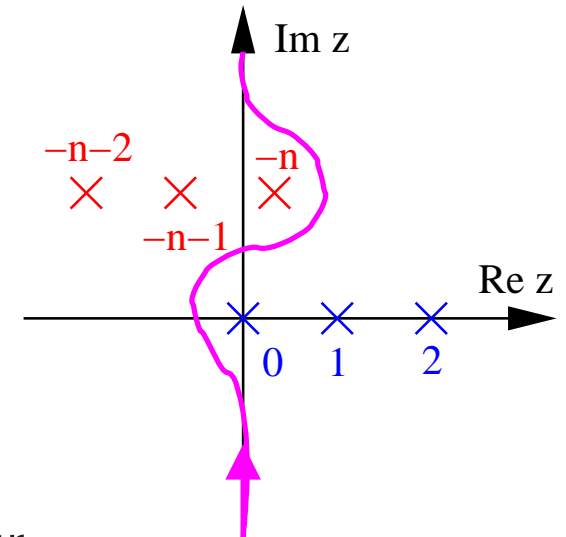
Ussyukina '75; Boos, Davydychev '91

Feynman integrals with many parameters are hard to evaluate

↪ separate parameters by Mellin–Barnes representation:

$$\frac{1}{(A+B)^n} = \frac{1}{\Gamma(n)} \int_{-i\infty}^{i\infty} \frac{dz}{2\pi i} \Gamma(-z) \Gamma(n+z) \frac{B^z}{A^{n+z}}$$

- Mellin–Barnes integrals go along the imaginary axis, leaving poles of $\Gamma(-z + \dots)$ to the right and poles of $\Gamma(z + \dots)$ to the left of the integration contour
- applicable to massive propagators ($A = k^2$, $B = -M^2$) or to any complicated parameter integral
- evaluation: close the integration contour to the right ($|B| \leq |A|$) or to the left ($|B| \geq |A|$) and pick up the residues within the contour: $\text{Res} \Gamma(z) \big|_{z=-i} = (-1)^i / i!$
- applicable for asymptotic expansions: sum of residues yields expansion in powers of (B/A) or (A/B) and $\ln(A/B)$
- close link to *expansion by regions*: Mellin–Barnes representation of the full integral → sum of residues
↪ asymptotic expansion with contributions corresponding to the regions



Mellin–Barnes integrals: extraction of singularities

$$I = \int_{-i\infty}^{i\infty} \frac{dz}{2\pi i} \underbrace{\Gamma(\alpha - z)}_{\text{right poles}} \underbrace{\Gamma(-\alpha + \epsilon + z)}_{\text{left poles}} f(z)$$

⇒ The **right pole** at $z = \alpha$ and the **left pole** at $z = \alpha - \epsilon$ “glue together” for $\epsilon \rightarrow 0$.

Close contour to the right:

$$-\text{Res} \Gamma(\alpha - z) \Gamma(-\alpha + \epsilon + z) f(z) \Big|_{z=\alpha} = \Gamma(\epsilon) f(\alpha) = \frac{1}{\epsilon} f(\alpha) + \mathcal{O}(\epsilon^0)$$

⇒ When a **left pole** and a **right pole** glue together, a singularity is produced!

Extraction of such singularities:

$$I = -\text{Res} \Gamma(\alpha - z) \Gamma(-\alpha + \epsilon + z) f(z) \Big|_{z=\alpha} + \int_{-i\infty}^{i\infty} \frac{dz}{2\pi i} \underbrace{\Gamma(1 + \alpha - z)}_{\text{right poles}} \underbrace{\frac{\Gamma(-\alpha + \epsilon + z)}{\alpha - z}}_{\text{left poles}} f(z)$$

Now the poles at $z = \alpha$ and $z = \alpha - \epsilon$ both lie to the left of the integration contour.

↪ The integrand can safely be expanded in ϵ .

New features with massive fermions

Expansion by regions with massive external legs

Expansion before integration:

1. expansion eliminates *small* terms with respect to *large* terms
2. integration produces new small, but possibly finite invariants

$$p_i^2, p_j^2 \in \{0, m_t^2\} \ll 2p_i \cdot p_j \sim Q^2$$

Need to make these invariants explicit *before integration*

$$\Rightarrow \text{shift to lightlike momenta: } p_{i,j} = \tilde{p}_{i,j} + \frac{p_{i,j}^2}{2\tilde{p}_i \cdot \tilde{p}_j} \tilde{p}_{j,i} \quad \text{with } \tilde{p}_i^2 = \tilde{p}_j^2 = 0$$

Relevant regions for each loop momentum k :

$$[M \sim M_{W,Z} \sim m_t \ll Q]$$

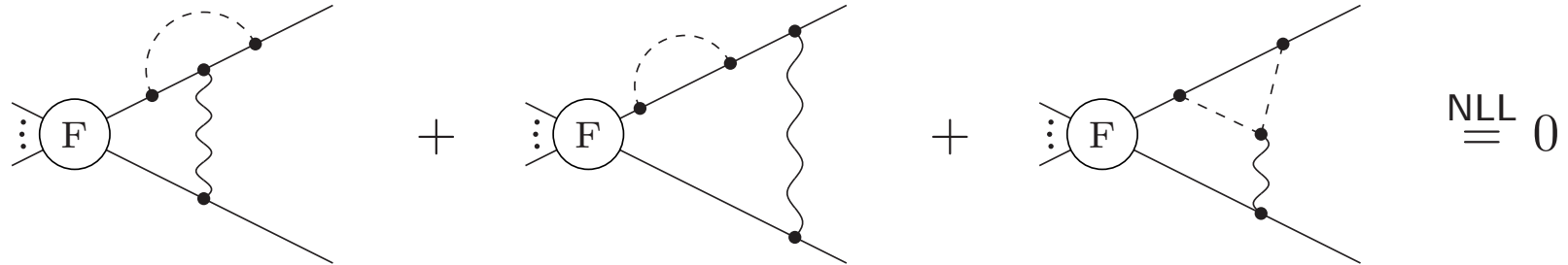
| | $k_{\parallel \tilde{p}_i}$ | $k_{\parallel \tilde{p}_j}$ | $k_{\perp(\tilde{p}_i, \tilde{p}_j)}$ |
|-----------|-----------------------------|-----------------------------|---------------------------------------|
| hard | Q | Q | Q |
| soft | M | M | M |
| ultrasoft | M^2/Q | M^2/Q | M^2/Q |

| | $k_{\parallel \tilde{p}_i}$ | $k_{\parallel \tilde{p}_j}$ | $k_{\perp(\tilde{p}_i, \tilde{p}_j)}$ | |
|--------------------------|-----------------------------|-----------------------------|---------------------------------------|------|
| <i>i</i> -collinear | Q | M^2/Q | M | |
| <i>j</i> -collinear | M^2/Q | Q | M | |
| <i>i</i> -ultracollinear | M^2/Q | M^4/Q^3 | M^3/Q^2 | new! |
| <i>j</i> -ultracollinear | M^4/Q^3 | M^2/Q | M^3/Q^2 | new! |

Yukawa contributions

Massive fermions \rightarrow Yukawa couplings to scalars (Higgs, Goldstone bosons)

- many Yukawa contributions are suppressed (soft/collinear limit, $M_W^2/Q^2 \rightarrow 0$)
- only three non-suppressed factorizable diagrams:



Sum vanishes due to **gauge invariance of Yukawa interaction**

\hookrightarrow NLL Yukawa contributions only in wave-function renormalization

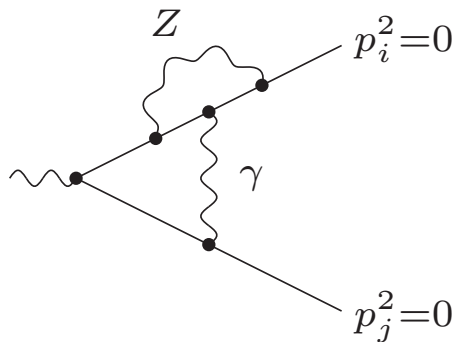
Power singularities Q^2/M^2

Asymptotic expansion for $Q^2 \gg M_{W,Z}^2, m_t^2$

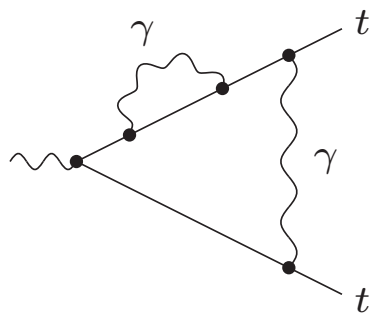
↪ logarithmic singularities $\ln(Q^2/M_{W,Z}^2), \ln(Q^2/m_t^2)$

↪ **power singularities** $Q^2/M_{W,Z}^2, Q^2/m_t^2$

E.g. in **scalar diagrams** (master integrals)



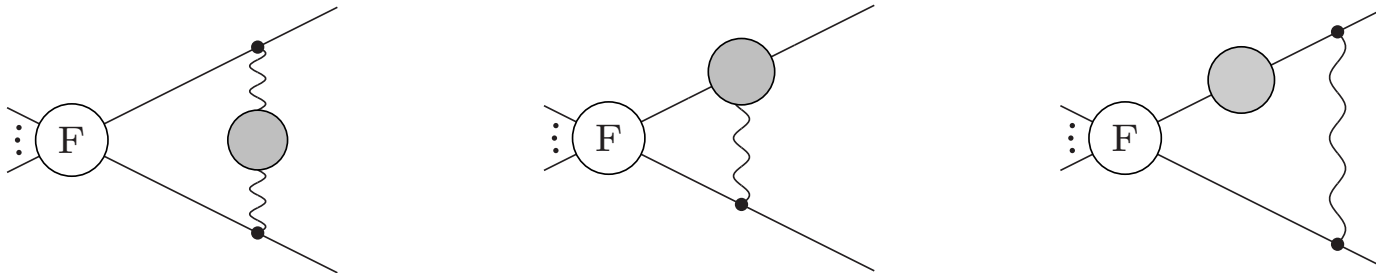
$$\stackrel{\text{NLL}}{=} -\frac{1}{Q^4} \frac{Q^2}{M_Z^2} \left(\frac{3}{4} \epsilon^{-3} + \frac{3}{2} L \epsilon^{-2} + \frac{3}{2} L^2 \epsilon^{-1} + L^3 \right)$$



$$\stackrel{\text{NLL}}{=} -\frac{1}{Q^4} \frac{Q^2}{m_t^2} \left(\frac{1}{4} L \epsilon^{-2} + \frac{3}{4} L^2 \epsilon^{-1} + \frac{7}{6} L^3 \right)$$

Power singularities Q^2/M^2 (2)

- not present at 1 loop
- appear at 2 loops in (scalar) diagrams with **loop insertions at soft-collinear lines**:



- **expansion by regions** predicts where power singularities can appear: simply combine the factors of M from propagators and integration measures for each region

Complete Feynman diagrams:

- Q^2/M^2 singularities always **compensated by factors of M^2/Q^2** from Feynman rules or reductions \Rightarrow results for Feynman diagrams are **free from power singularities**
- **massless fermions**: power singularities do not affect fermion lines
- **massive fermions**: **mass terms in numerator of fermion lines important!**
 - \hookrightarrow soft-collinear approximation not possible for the subdiagrams from above
 - \Rightarrow treat these subdiagrams with projection techniques

Power singularities: complications from fermion masses

Mass terms in numerator of fermion lines important:

- fermion propagators: $\frac{\not{k} + m_t}{k^2 - m_t^2}$
- spinors: $(\not{p} - m_t) u(p) = 0$, $(\not{p} + m_t) v(p) = 0$

⇒ Need **more complicated projection** on Dirac structure than in massless case:

$$\mathcal{M} = G_0 \Gamma(p_i, p_j) u(p_i) = \Gamma_1 \underbrace{G_0 u(p_i)}_{\text{Born}} + \underbrace{\Gamma_2 G_0 \not{p}_j u(p_i)}_{\text{suppressed for } Q^2 \gg m_t^2}$$

⇒ **Spin eigenstates** $u(p, \sigma)$ are not exact eigenstates of the **chirality projectors**

$$\omega_{R,L} = \frac{1}{2}(1 \pm \gamma^5).$$

⇒ One heavy-quark line may involve EW couplings with **different chiralities**

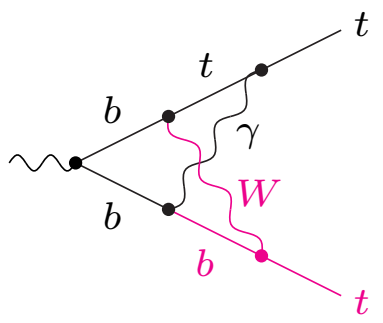
↪ happens only with QED couplings which are independent of chirality.

Structure of the corrections: new logarithms

Scale $\Delta \sim M_W^2 \ll Q^2$ of soft-collinear contributions \rightarrow logs

$$\ln^n \left(\frac{Q^2}{\Delta} \right) \stackrel{\text{NLL}}{=} \ln^n \left(\frac{Q^2}{M_W^2} \right) - n \ln \left(\frac{\Delta}{M_W^2} \right) \ln^{n-1} \left(\frac{Q^2}{M_W^2} \right)$$

- massless fermions: only $\Delta = M_{Z,W}^2$
- massive fermions: also $\Delta = m_t^2$ and $\Delta = M_W^2 - m_t^2 - i0$ (at W - t - b vertices):



| | | |
|--------------------|-----------|---|
| scalar diagram | \propto | $\frac{5}{6}L^4 - \frac{4}{3} \left[\ln \left(\frac{m_t^2}{M_W^2} \right) + \ln \left(\frac{M_W^2 - m_t^2}{M_W^2} \right) \right] L^3$ |
| Feynman diagram | \propto | $\frac{2}{3}L^4 - \left[4 + \frac{2}{3} \ln \left(\frac{m_t^2}{M_W^2} \right) \right] L^3$ |

Complete Feynman diagrams:

- diagrams **without photons**: independent of fermion masses, only $\Delta = M_{Z,W}^2$
- diagrams **with photons**: $\Delta = M_{Z,W}^2$ and $\Delta = m_t^2$, but not $\Delta = M_W^2 - m_t^2$
(additionally $1/\epsilon$ poles)

Overview

I Electroweak (EW) corrections at high energies

- origin and importance of EW logs
- existing virtual EW 2-loop corrections

II Two-loop next-to-leading logarithmic (NLL) corrections

- extraction of NLL logs
- evaluation of Feynman diagrams:
expansion by regions & Mellin–Barnes representation
- new features with massive fermions

III Results for processes involving massless and massive fermions

- factorization and exponentiation of the logs
- comparisons and applications
- structure of the result to all orders in ϵ

IV Summary & outlook

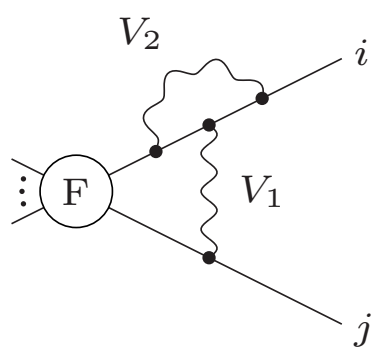
III Results for processes involving massless and massive fermions

Factorizable contributions

loop integrals calculated with two independent methods:

- automatized algorithm based on **sector decomposition** Denner, Pozzorini '04
- combination of **expansion by regions** & **Mellin–Barnes representations** B.J., Smirnov '06 & refs. therein

Example for the contribution of a Feynman diagram



$$\begin{array}{c}
 \text{Born amplitude} \\
 \text{factorized} \\
 \uparrow \\
 \mathcal{M}_0
 \end{array}
 \underbrace{I_i^{V_2} I_i^{V_1} I_i^{\bar{V}_2} I_j^{\bar{V}_1}}_{\text{isospin matrices @ external legs}}
 \underbrace{D(M_{V_1}, M_{V_2}; p_i, p_j)}_{\text{scalar 2-loop integral}}
 \sum_{\substack{V_1, V_2 = \gamma, Z, W^\pm \\ \text{sum over gauge bosons}}} \stackrel{\text{NLL}}{=}$$

All relevant combinations of $\left\{ \begin{array}{l} \text{massless} \\ \text{massive} \end{array} \right\}$ $\left\{ \begin{array}{l} \text{external} \\ \text{internal} \end{array} \right\}$ fermions evaluated explicitly!

Result for amplitude of fermionic processes $f_1 f_2 \rightarrow f_3 \cdots f_n$ in $\mathcal{O}(\alpha^2)$

$$\mathcal{M} \stackrel{\text{NLL}}{=} \underbrace{\exp(\Delta F^{\text{em}})}_{\substack{\text{electromagnetic} \\ M_\gamma = 0}} \times \underbrace{\exp(F^{\text{sew}})}_{\substack{\text{symmetric-electroweak} \\ M_\gamma = M_Z = M_W}} \times \underbrace{(1 + \Delta F^Z)}_{\substack{\text{corrections} \\ \text{from } M_Z \neq M_W}} \times \underbrace{\mathcal{M}_0}_{\text{Born}}$$

- **universal** result: F^{sew} , ΔF^{em} , ΔF^Z depend only on external quantum numbers
- electromagnetic singularities (in ΔF^{em}) factorized \rightarrow separable

Symmetric-electroweak terms: independent of fermion masses

$$F^{\text{sew}} = \frac{1}{2} \sum_{i=1}^n \left\{ -\frac{\alpha}{4\pi} \left[\sum_{j \neq i} \sum_{V=\gamma, Z, W^\pm} I_i^{\bar{V}} I_j^V I_{ij}(\epsilon, M_W) + \overbrace{\frac{z_i^{\text{Yuk}} m_t^2}{4s_W^2 M_W^2} \left(L + \frac{1}{2} L^2 \epsilon + \frac{1}{6} L^3 \epsilon^2 \right)}^{\text{Yukawa contribution}} \right] \right. \\ \left. + \left(\frac{\alpha}{4\pi} \right)^2 \left[\frac{b_1^{(1)}}{c_W^2} \left(\frac{Y_i}{2} \right)^2 + \frac{b_2^{(1)}}{s_W^2} C_i^W \right] J_{ii}(\epsilon, M_W, \mu_R^2) \right\},$$

$$I_{ij}(\epsilon, M_W) \stackrel{\text{NLL}}{=} -L^2 - \frac{2}{3} L^3 \epsilon - \frac{1}{4} L^4 \epsilon^2 + \left[3 - 2 \ln \left(\frac{-r_{ij}}{Q^2} \right) \right] \left(L + \frac{1}{2} L^2 \epsilon + \frac{1}{6} L^3 \epsilon^2 \right) + \mathcal{O}(\epsilon^3),$$

$$J_{ij}(\epsilon, M_W, \mu^2) = \frac{1}{\epsilon} \left[I_{ij}(2\epsilon, M_W) - \left(\frac{Q^2}{\mu^2} \right)^\epsilon I_{ij}(\epsilon, M_W) \right]$$

$$\mathcal{M} \stackrel{\text{NLL}}{=} \exp(\Delta F^{\text{em}}) \times \exp(F^{\text{sew}}) \times (1 + \Delta F^{\text{Z}}) \times \mathcal{M}_0$$

Electromagnetic terms: QED($M_\gamma = 0$) – QED($M_\gamma = M_W$) [$\mu_R^2 = M_W^2$]

$$\Delta F^{\text{em}} = \frac{1}{2} \sum_{i=1}^n \left\{ -\frac{\alpha}{4\pi} \sum_{j \neq i} Q_i Q_j \left[I_{ij}(\epsilon, 0) - I_{ij}(\epsilon, M_W) \right] + \left(\frac{\alpha}{4\pi} \right)^2 b_{\text{QED}}^{(1)} Q_i^2 \left[J_{ii}(\epsilon, 0, M_W^2) - J_{ii}(\epsilon, M_W, M_W^2) \right] \right\},$$

dependence on fermion mass m_i

$$I_{ij}(\epsilon, 0) \stackrel{\text{NLL}}{=} - \left[3 - 2 \ln \left(\frac{-r_{ij}}{Q^2} \right) \right] \epsilon^{-1} + \left\{ \overbrace{-\delta_{i,0} \epsilon^{-2} + \delta_{i,t}} \left[L \epsilon^{-1} + \frac{1}{2} L^2 + \frac{1}{6} L^3 \epsilon + \frac{1}{24} L^4 \epsilon^2 \right] + \left(\frac{1}{2} - \ln \left(\frac{m_i^2}{M_W^2} \right) \right) \left(\epsilon^{-1} + L + \frac{1}{2} L^2 \epsilon + \frac{1}{6} L^3 \epsilon^2 \right) \right\} + (i \rightarrow j) \left. \right\} + \mathcal{O}(\epsilon^3),$$

$$J_{ij}(\epsilon, 0, \mu^2) = \frac{1}{\epsilon} \left[I_{ij}(2\epsilon, 0) - \left(\frac{Q^2}{\mu^2} \right)^\epsilon I_{ij}(\epsilon, 0) \right]$$

Terms from $M_Z \neq M_W$:

$$\Delta F^{\text{Z}} \stackrel{\text{NLL}}{=} \frac{\alpha}{4\pi} \sum_{i=1}^n (I_i^{\text{Z}})^2 \overbrace{\ln \left(\frac{M_Z^2}{M_W^2} \right) \left(L + L^2 \epsilon + \frac{1}{2} L^3 \epsilon^2 \right)}^{= I_{ii}(\epsilon, M_Z) - I_{ii}(\epsilon, M_W)} + \mathcal{O}(\epsilon^3)$$

Comparisons and applications

Comparison to existing results

- previous results for **form factor** and **angular-dependent NLLs** reproduced and extended Denner, Melles, Pozzorini '03; Pozzorini '04
- structure of symmetric-electroweak NLL corrections in complete analogy with **Catani's formula for massless QCD** Catani '98
- agreement with general **resummation predictions** based on evolution equations Melles '00, '01
- agreement with **SCET results** Chiu, Golf, Kelley, Manohar '07, '08

Application to 4-fermion scattering

- **neutral current** $f\bar{f} \rightarrow f'\bar{f}'$: NLL-agreement with massless-fermion N³LL calculation, additional fermion-mass effects B.J., Kühn, Penin, Smirnov '05
- **charged current** $f_1\bar{f}_2 \rightarrow f_3\bar{f}_4$: new NLL result
- also applicable to processes with fermions and **gluons**, e.g. $g g \rightarrow f\bar{f}$: external SU(2) × U(1) singlets \rightsquigarrow legs with zero quantum numbers

Structure of the result to all orders in ϵ

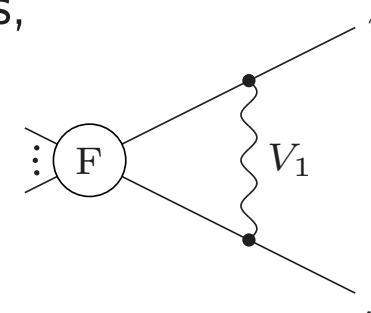
Expansion by regions: look at contributions from individual regions, e.g. in 1-loop diagram (with minimal UV subtraction):

hard region: $-2 (\epsilon^{-2} + 2\epsilon^{-1}) \left(\frac{Q^2}{-r_{ij}} \right)^\epsilon$

collinear regions:

- $m_1 = 0 \Rightarrow (\epsilon^{-2} + 2\epsilon^{-1}) \left[\left(\frac{Q^2}{m_i^2} \right)^\epsilon + \left(\frac{Q^2}{m_j^2} \right)^\epsilon \right]$

- $m_1 \neq 0 \Rightarrow 2 \left[\epsilon^{-2} - \underbrace{\ln \left(\frac{-r_{ij}}{m_1^2} \right)}_{\text{finite remainder from singularity cancelled between i-/j-collinear regions}} \epsilon^{-1} + 2\epsilon^{-1} \right] \left(\frac{Q^2}{m_1^2} \right)^\epsilon$



Each region depends on mass parameters via **one unique power** of $(Q^2/m^2)^\epsilon$.

\hookrightarrow Logs $\ln(Q^2/m^2)$ are generated by poles ϵ^{-n} in prefactor.

\hookrightarrow Additional logarithms arise from singularities cancelled between collinear regions.

\hookrightarrow $\mathcal{O}(\epsilon^0)$ in prefactor is beyond NLL accuracy.

\rightsquigarrow In NLL accuracy this representation is valid **to all orders in ϵ** !

NLL result to all orders in ϵ

$$\mathcal{M} \stackrel{\text{NLL}}{=} \exp(\Delta F^{\text{em}}) \times \exp(F^{\text{sew}}) \times (1 + \Delta F^{\text{Z}}) \times \mathcal{M}_0$$

Every part of the result is known to all orders in ϵ . Most important ingredient:

$$I_{ij}(\epsilon, m_1) \stackrel{\text{NLL}}{=} - (2\epsilon^{-2} + 3\epsilon^{-1}) z_{ij}^{-\epsilon} + \left\{ \left[2\epsilon^{-2} - 2L\epsilon^{-1} + (3 - 2l_{ij} + 2l_1) \epsilon^{-1} \right] z_1^{-\epsilon} + \delta_{1,\gamma} \left(\epsilon^{-2} + \frac{1}{2}\epsilon^{-1} \right) (z_i^{-\epsilon} + z_j^{-\epsilon}) \right\} Z^\epsilon$$

with

$$Z = \frac{Q^2}{M_W^2}, \quad L = \ln Z; \quad z_{ij} = \frac{-r_{ij}}{Q^2}, \quad l_{ij} = \ln z_{ij}; \quad z_a = \frac{m_a^2}{M_W^2}, \quad l_a = \ln z_a, \quad a = 1, 2, \dots, i, j, \dots$$

$$z_a^{n\epsilon} \equiv 0 \text{ if } m_a = 0, \quad n \neq 0; \quad \delta_{a,\gamma} = \begin{cases} 1, & m_a = 0 \\ 0, & m_a \sim M_W \end{cases}$$

Exponentiation

1-loop $\rightsquigarrow I_{ij}(\epsilon, m_1)$: Z^0 (hard), Z^ϵ (collinear)

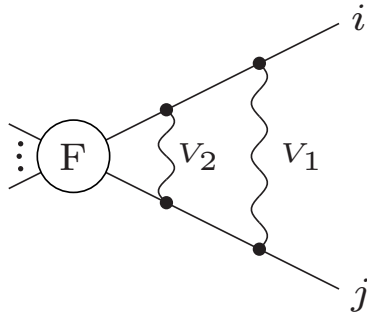
2-loop $\rightsquigarrow I_{ij}(\epsilon, m_1) \times I_{kl}(\epsilon, m_2), I_{ij}(2\epsilon, m_1), Z^\epsilon \times I_{ij}(\epsilon, m_1)$:

Z^0 (hard–hard), Z^ϵ (hard–collinear), $Z^{2\epsilon}$ (collinear–collinear)

2-loop contributions to all orders in ϵ

2-loop result involves Z^0 (hard-hard), Z^ϵ (hard-collinear), $Z^{2\epsilon}$ (collinear-collinear)

But:

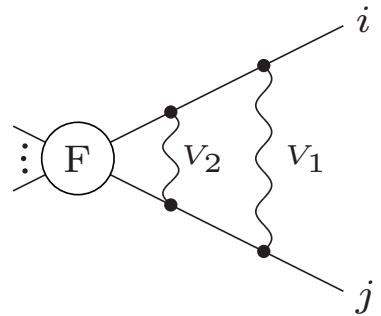


$$\begin{aligned}
 &\hookrightarrow (\epsilon^{-4} + 4\epsilon^{-3}) z_{ij}^{-2\epsilon} \\
 &+ \left\{ 4 \left[\epsilon^{-4} + L\epsilon^{-3} + (l_{ij} - l_1) \epsilon^{-3} + 2L\epsilon^{-2} \right] z_1^{-\epsilon} - \frac{2}{3} \delta_{1,\gamma} (\epsilon^{-4} + 4\epsilon^{-3}) (z_i^{-\epsilon} + z_j^{-\epsilon}) \right\} z_{ij}^{-\epsilon} Z^\epsilon \\
 &+ \left\{ - \left[5\epsilon^{-4} - 2L\epsilon^{-3} + 2(2 - l_{ij} + l_1) \epsilon^{-3} \right] z_1^{-2\epsilon} \right. \\
 &\quad \left. + \delta_{1,\gamma} \left[- \left(\epsilon^{-4} - 2L\epsilon^{-3} + 2(2 - l_{ij} + l_2) \epsilon^{-3} \right) z_2^{-2\epsilon} + (\epsilon^{-4} + 2\epsilon^{-3}) z_2^{-\epsilon} (z_i^{-\epsilon} + z_j^{-\epsilon}) \right] \right\} Z^{2\epsilon} \\
 &+ \delta_{1,\gamma} \left\{ -\frac{1}{3} (\epsilon^{-4} - 2\epsilon^{-3}) z_2^{-3\epsilon} (z_i^{-\epsilon} + z_j^{-\epsilon}) + \frac{1}{6} \delta_{2,\gamma} (\epsilon^{-4} + 4\epsilon^{-3}) (z_i^{-\epsilon} z_j^{-3\epsilon} + z_j^{-\epsilon} z_i^{-3\epsilon}) \right\} z_{ij}^{2\epsilon} Z^{4\epsilon}
 \end{aligned}$$

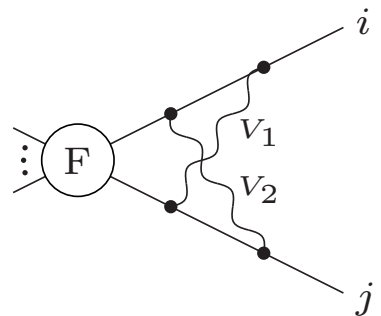
new contribution: $Z^{4\epsilon}$ (ultracollinear-collinear), not present in 2-loop result?!

\hookrightarrow need non-trivial cancellation of all $Z^{4\epsilon}$ terms in total amplitude!

Cancellation of $Z^{4\epsilon}$ terms in 2-loop result

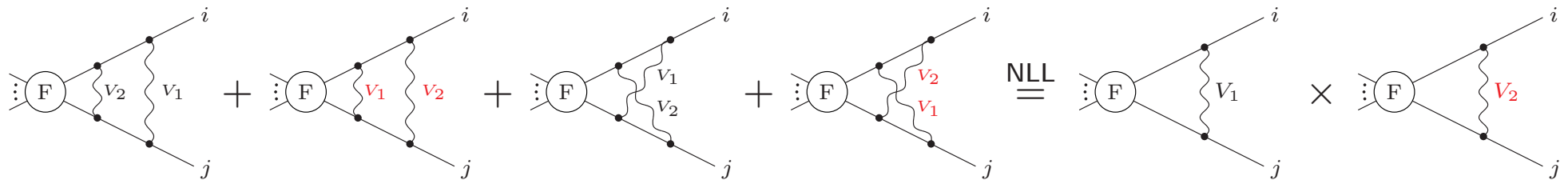


$$\delta_{1,\gamma} \left\{ -\frac{1}{3} (\epsilon^{-4} - 2\epsilon^{-3}) z_2^{-3\epsilon} (z_i^{-\epsilon} + z_j^{-\epsilon}) + \frac{1}{6} \delta_{2,\gamma} (\epsilon^{-4} + 4\epsilon^{-3}) (z_i^{-\epsilon} z_j^{-3\epsilon} + z_j^{-\epsilon} z_i^{-3\epsilon}) \right\} z_{ij}^{2\epsilon} Z^{4\epsilon}$$



$$\left\{ \frac{1}{3} (\epsilon^{-4} - 2\epsilon^{-3}) (\delta_{1,\gamma} z_2^{-3\epsilon} z_i^{-\epsilon} + \delta_{2,\gamma} z_1^{-3\epsilon} z_j^{-\epsilon}) - \frac{1}{6} \delta_{1,\gamma} \delta_{2,\gamma} (\epsilon^{-4} + 4\epsilon^{-3}) (z_i^{-\epsilon} z_j^{-3\epsilon} + z_j^{-\epsilon} z_i^{-3\epsilon}) \right\} z_{ij}^{2\epsilon} Z^{4\epsilon}$$

$\Rightarrow Z^{4\epsilon}$ terms cancel in combination of scalar loop integrals:



\hookrightarrow this relation (and others) checked to all orders in ϵ \checkmark

Overview

I Electroweak (EW) corrections at high energies

- origin and importance of EW logs
- existing virtual EW 2-loop corrections

II Two-loop next-to-leading logarithmic (NLL) corrections

- extraction of NLL logs
- evaluation of Feynman diagrams:
expansion by regions & Mellin–Barnes representation
- new features with massive fermions

III Results for processes involving massless and massive fermions

- factorization and exponentiation of the logs
- comparisons and applications
- structure of the result to all orders in ϵ

IV Summary & outlook

IV Summary & outlook

Massless and massive fermionic processes $f_1 f_2 \rightarrow f_3 \cdots f_n$ (+ gluons)

with $(p_i + p_j)^2 \gg M_W^2$ and different masses $M_W^2 \sim M_Z^2 \sim m_t^2 \sim M_{\text{Higgs}}^2$:

- **2-loop EW NLL corrections** in $D = 4 - 2\epsilon$ dimensions
- loop integrals calculated with two independent methods, expansion by regions & Mellin–Barnes representations presented here
- Yukawa contributions only in wave-function renormalization
- **universal correction factors**, electromagnetic singularities separable
- applicable for $e^+ e^- \rightarrow f \bar{f}$, Drell–Yan, $gg \rightarrow f \bar{f}$, ...
- NLL result available **to all orders in ϵ** \rightarrow structure with powers of $(Q^2/M_W^2)^\epsilon$
 \hookrightarrow non-trivial cancellations ensure exponentiation of 1-loop result

Outlook: EW corrections to arbitrary high-energy processes

- generalize method for external gauge bosons & scalars (Higgs)
- calculate relevant loop integrals
- goal: **process-independent 2-loop NLL corrections**

Extra slides

Expansion by regions: how it works

simple 1-dimensional example: $f = \int_0^\infty \frac{dk k^{-\epsilon}}{(k+m)(k+q)}, \quad m \ll q$

$$\left. \begin{array}{l} \text{soft (s): } k \sim m, \quad k < \Lambda \\ \text{hard (h): } k \sim q, \quad k > \Lambda \end{array} \right\} \text{ where } m \ll \Lambda \ll q$$

$$\begin{aligned} f &= \int_0^\Lambda \frac{dk k^{-\epsilon}}{(k+m)(k+q)} + \int_\Lambda^\infty \frac{dk k^{-\epsilon}}{(k+m)(k+q)} \\ &= \sum_{j=0}^{\infty} \frac{(-1)^j}{q^{j+1}} \int_0^\Lambda \frac{dk k^{-\epsilon+j}}{k+m} + \sum_{i=0}^{\infty} (-m)^i \int_\Lambda^\infty \frac{dk k^{-\epsilon-i-1}}{k+q} \\ &= \sum_{j=0}^{\infty} \frac{(-1)^j}{q^{j+1}} \left(\int_0^\infty \frac{dk k^{-\epsilon+j}}{k+m} - \int_\Lambda^\infty \frac{dk k^{-\epsilon+j}}{k+m} \right) + \sum_{i=0}^{\infty} (-m)^i \left(\int_0^\infty \frac{dk k^{-\epsilon-i-1}}{k+q} - \int_0^\Lambda \frac{dk k^{-\epsilon-i-1}}{k+q} \right) \\ &= \underbrace{\sum_{j=0}^{\infty} \frac{(-1)^j}{q^{j+1}} \int_0^\infty \frac{dk k^{-\epsilon+j}}{k+m}}_{f(s)} + \underbrace{\sum_{i=0}^{\infty} (-m)^i \int_0^\infty \frac{dk k^{-\epsilon-i-1}}{k+q}}_{f(h)} - \sum_{i=0}^{\infty} (-m)^i \sum_{j=0}^{\infty} \frac{(-1)^j}{q^{j+1}} \underbrace{\int_0^\infty dk k^{-\epsilon-i+j-1}}_{\rightarrow 0, \text{ scaleless integral}} \\ &= f(s) + f(h) = \frac{\Gamma(\epsilon)\Gamma(1-\epsilon)}{m^\epsilon q} \sum_{j=0}^{\infty} \left(\frac{m}{q}\right)^j + \frac{\Gamma(-\epsilon)\Gamma(1+\epsilon)}{q^{1+\epsilon}} \sum_{i=0}^{\infty} \left(\frac{m}{q}\right)^i \\ &= \frac{\Gamma(\epsilon)\Gamma(1-\epsilon)}{(q-m)m^\epsilon} + \frac{\Gamma(-\epsilon)\Gamma(1+\epsilon)}{(q-m)q^\epsilon} = \frac{\ln(q/m)}{q-m} + \mathcal{O}(\epsilon) \quad \checkmark \end{aligned}$$

Parameterization of Feynman integrals

- Schwinger parameters:

$$\frac{1}{A^n} = \frac{1}{i^n \Gamma(n)} \int_0^\infty d\alpha \alpha^{n-1} e^{i\alpha A}, \quad \text{numerator } A^n = \left(\frac{1}{i} \frac{\partial}{\partial \alpha} \right)^n e^{i\alpha A} \Big|_{\alpha=0}$$

⇒ any number of propagators and numerators may be combined

⇒ can always be transformed to (generalized) Feynman parameters

↪ evaluation:

$$\int d^d k e^{i(\alpha k^2 + 2p \cdot k)} = i\pi^{d/2} (i\alpha)^{-d/2} e^{-ip^2/\alpha}$$

$$\int_0^\infty \frac{d\alpha \alpha^{n-1}}{(A + \alpha B)^r} = \frac{\Gamma(n) \Gamma(r-n)}{\Gamma(r) A^{r-n} B^n}$$

- generalized Feynman parameters:

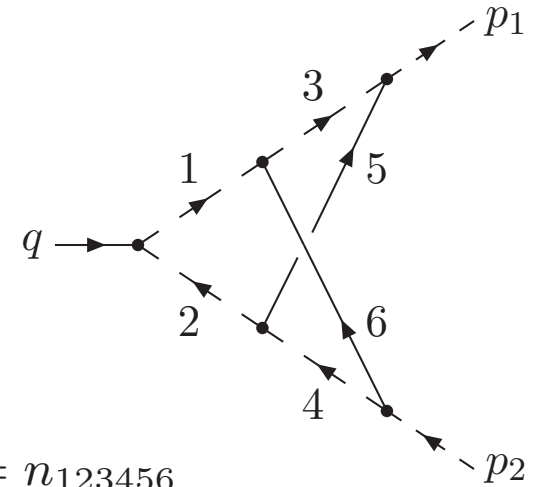
$$\prod_{i=1}^L \frac{1}{A_i^{n_i}} = \frac{\Gamma(\sum_i n_i)}{\prod_i \Gamma(n_i)} \left(\prod_i \int_0^\infty dx_i x_i^{n_i-1} \right) \frac{\delta\left(\sum_{j \in S} x_j - 1\right)}{(\sum_i x_i A_i)^{\sum_i n_i}}, \quad \emptyset \neq S \subseteq \{1, \dots, L\}$$

⇒ convenient also for non-standard propagators, e.g. $A_i = 2p \cdot k$

Example: the non-planar vertex diagram

Scalar integrals with variable powers of propagators:

$$\begin{aligned}
 F_{\text{NP}}(n_1, \dots, n_7) &= e^{2\epsilon\gamma} (M^2)^{2\epsilon} (Q^2)^{n-n_7-4} \\
 &\times \int \frac{d^d k}{i\pi^{d/2}} \int \frac{d^d \ell}{i\pi^{d/2}} \frac{(2k \cdot \ell)^{n_7}}{((p_1 - k - \ell)^2)^{n_1} ((p_2 - k - \ell)^2)^{n_2}} \\
 &\times \frac{1}{(k^2 - 2p_1 \cdot k)^{n_3} (\ell^2 - 2p_2 \cdot \ell)^{n_4} (k^2 - M^2)^{n_5} (\ell^2 - M^2)^{n_6}}, \quad n = n_{123456}
 \end{aligned}$$



Contributing regions: (h-h), (1c-h), (1c-1c), (1c-2c), (1c-1c'), (1c-us'), (h-2c), (2c-2c), (2c'-2c), (us'-us'), (us'-2c).

Leading term of (1c-h) region $\iff k^2 \sim 2p_1 \cdot k \sim M^2, 2p_2 \cdot k \sim Q^2, \ell \sim Q$:

$$\begin{aligned}
 F_{\text{NP}}^{(1c-h)}(n_1, \dots, n_7) &= e^{2\epsilon\gamma} (M^2)^{2\epsilon} (Q^2)^{n-n_7-4} \int \frac{d^d k}{i\pi^{d/2}} \int \frac{d^d \ell}{i\pi^{d/2}} \\
 &\times \frac{((2p_2 \cdot k)(2p_1 \cdot \ell)/Q^2)^{n_7}}{(\ell^2 - 2p_1 \cdot \ell + (2p_2 \cdot k)(2p_1 \cdot \ell)/Q^2)^{n_1} (\ell^2 - 2p_2 \cdot (k + \ell) + (2p_2 \cdot k)(2p_1 \cdot \ell)/Q^2)^{n_2}} \\
 &\times \frac{1}{(k^2 - 2p_1 \cdot k)^{n_3} (\ell^2 - 2p_2 \cdot \ell)^{n_4} (k^2 - M^2)^{n_5} (\ell^2)^{n_6}} + \mathcal{O}\left(\frac{M^2}{Q^2}\right)
 \end{aligned}$$

Example: (1c-h) region of the non-planar vertex diagram

Introduce **Feynman or Schwinger parameters**, integrate & transform into

$$F_{\text{NP}}^{(1\text{c-h})}(n_1, \dots, n_7) = \left(\frac{M^2}{Q^2}\right)^{2-n_{35}+\epsilon} (-1)^n e^{2\epsilon\gamma} \frac{\Gamma(\frac{d}{2} - n_{24})\Gamma(\frac{d}{2} - n_{16} + n_7)\Gamma(n_{35} - \frac{d}{2})}{\Gamma(d - n_{1246} + n_7) \prod_{i=1}^6 \Gamma(n_i)}$$

$$\times \int_0^1 dx_1 dx_2 dx_3 x_1^{n_1-1} (1-x_1)^{n_6-1} x_2^{n_2-1} (1-x_2)^{n_4-1} x_3^{n_3-1} (1-x_3)^{\frac{d}{2}-n_3-1}$$

$$\times \underbrace{\Gamma(n_{1246} - \frac{d}{2}) [x_1(1-x_3) + x_2x_3]^{\frac{d}{2}-n_{1246}}}_{\text{Mellin-Barnes representation:}}$$

$$\int_{-i\infty}^{i\infty} \frac{dz}{2\pi i} \Gamma(-z)\Gamma(n_{1246} - \frac{d}{2} + z) (x_1(1-x_3))^z (x_2x_3)^{\frac{d}{2}-n_{1246}-z}$$

⇒ Expression with Mellin-Barnes integral:

$$F_{\text{NP}}^{(1\text{c-h})}(n_1, \dots, n_7) = \left(\frac{M^2}{Q^2}\right)^{2-n_{35}+\epsilon} (-1)^n \frac{e^{2\epsilon\gamma} \Gamma(\frac{d}{2} - n_{24})\Gamma(\frac{d}{2} - n_{16} + n_7)\Gamma(n_{35} - \frac{d}{2})}{\Gamma(n_1)\Gamma(n_2)\Gamma(n_3)\Gamma(n_5) \Gamma(d - n_{1246} + n_7)^2}$$

$$\times \int_{-i\infty}^{i\infty} \frac{dz}{2\pi i} \frac{\Gamma(-z)\Gamma(\frac{d}{2} - n_{146} - z)\Gamma(\frac{d}{2} - n_{1246} + n_{37} - z)}{\Gamma(\frac{d}{2} - n_{16} - z)}$$

$$\times \frac{\Gamma(n_1 + z)\Gamma(\frac{d}{2} - n_3 + z)\Gamma(n_{1246} - \frac{d}{2} + z)}{\Gamma(n_{16} + z)}$$

Example: evaluation of the (1c-h) region for special cases (1)

$$F_{\text{NP}}^{(1\text{c-h})}(1, 1, 1, 0, 1, 0, 0) = \left(\frac{M^2}{Q^2}\right)^\epsilon e^{2\epsilon\gamma} \frac{\Gamma(1-\epsilon)^2 \Gamma(\epsilon)}{\Gamma(2-2\epsilon)^2} \\ \times \int_{-i\infty}^{i\infty} \frac{dz}{2\pi i} \Gamma(-z) \Gamma(1-\epsilon-z) \Gamma(1-\epsilon+z) \Gamma(\epsilon+z)$$

Solution known: 1st Barnes lemma

Barnes 1908

$$\int_{-i\infty}^{i\infty} \frac{dz}{2\pi i} \Gamma(\alpha_1 - z) \Gamma(\alpha_2 - z) \Gamma(\alpha_3 + z) \Gamma(\alpha_4 + z) = \\ \frac{\Gamma(\alpha_1 + \alpha_3) \Gamma(\alpha_1 + \alpha_4) \Gamma(\alpha_2 + \alpha_3) \Gamma(\alpha_2 + \alpha_4)}{\Gamma(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4)}$$

$$\Rightarrow F_{\text{NP}}^{(1\text{c-h})}(1, 1, 1, 0, 1, 0, 0) = \left(\frac{M^2}{Q^2}\right)^\epsilon e^{2\epsilon\gamma} \frac{\Gamma(1-\epsilon)^2 \Gamma(\epsilon)}{\Gamma(2-2\epsilon)^2} \frac{\Gamma(1-\epsilon) \Gamma(\epsilon) \Gamma(2-2\epsilon)}{\Gamma(2-\epsilon)} \\ = \frac{1}{\epsilon^2} + \frac{1}{\epsilon} (-L + 3) + \frac{1}{2} L^2 - 3L + 7 + \mathcal{O}(\epsilon), \quad L = \ln\left(\frac{Q^2}{M^2}\right)$$

Example: evaluation of the (1c-h) region for special cases (2)

$$F_{\text{NP}}^{(1\text{c-h})}(1, 1, 1, 0, 1, 1, 0) = - \left(\frac{M^2}{Q^2} \right)^\epsilon e^{2\epsilon\gamma} \frac{\Gamma(1-\epsilon)\Gamma(-\epsilon)\Gamma(\epsilon)}{\Gamma(1-2\epsilon)^2} \\ \times \int_{-i\infty}^{i\infty} \frac{dz}{2\pi i} \Gamma(-z)\Gamma(-\epsilon-z) \frac{\Gamma(1+z)\Gamma(1-\epsilon+z)\Gamma(1+\epsilon+z)}{\Gamma(2+z)}$$

- 1st possibility: expand integrand in ϵ and/or cancel functions in denominator
 \hookrightarrow transform to expressions solvable by Barnes lemma etc.
- 2nd possibility: close integration contour to the right and take residues directly:

$$\left(\frac{M^2}{Q^2} \right)^\epsilon e^{2\epsilon\gamma} \frac{\Gamma(1-\epsilon)^3\Gamma(1+\epsilon)^2}{\epsilon^3\Gamma(1-2\epsilon)^2} \sum_{i=0}^{\infty} \left[\underbrace{\frac{\Gamma(1-2\epsilon+i)}{\Gamma(2-\epsilon+i)}}_{\text{from } z=-\epsilon+i} - \underbrace{\frac{\Gamma(1-\epsilon+i)}{\Gamma(2+i)}}_{\text{from } z=i} \right]$$

expand Gamma functions \rightarrow sum up to (multiple) zeta values:

$$F_{\text{NP}}^{(1\text{c-h})}(1, 1, 1, 0, 1, 1, 0) = \frac{\pi^2}{6\epsilon^2} + \frac{1}{\epsilon} \left(-\frac{\pi^2}{6}L + 2\zeta_3 \right) + \frac{\pi^2}{12}L^2 - 2\zeta_3L + \frac{\pi^4}{40} + \mathcal{O}(\epsilon)$$

Example: evaluation of the (1c-h) region for special cases (3)

$$F_{\text{NP}}^{(1\text{c-h})}(1, \delta, 1, 1, 1, 1, 0) = - \left(\frac{M^2}{Q^2} \right)^\epsilon (-1)^\delta e^{2\epsilon\gamma} \frac{\Gamma(1 - \epsilon - \delta)\Gamma(-\epsilon)\Gamma(\epsilon)}{\Gamma(\delta)\Gamma(1 - 2\epsilon - \delta)^2}$$

$$\times \int_{-i\infty}^{i\infty} \frac{dz}{2\pi i} \frac{\Gamma(-z)\Gamma(-1 - \epsilon - z)\Gamma(-\epsilon - \delta - z)}{\Gamma(-\epsilon - z)} \frac{\Gamma(1 + z)\Gamma(1 - \epsilon + z)\Gamma(1 + \epsilon + \delta + z)}{\Gamma(2 + z)}$$

Limit $\delta \rightarrow 0 \Rightarrow \frac{1}{\Gamma(\delta)} \rightarrow 0$, but gluing poles at $z = -1 - \epsilon$ and $z = -1 - \epsilon - \delta$:

$$F_{\text{NP}}^{(1\text{c-h})}(1, 0, 1, 1, 1, 1, 0) = \lim_{\delta \rightarrow 0} \left(\frac{M^2}{Q^2} \right)^\epsilon (-1)^\delta e^{2\epsilon\gamma} \frac{\Gamma(1 - \epsilon - \delta)\Gamma(-\epsilon)\Gamma(\epsilon)}{\Gamma(\delta)\Gamma(1 - 2\epsilon - \delta)^2}$$

$$\times \text{Res} \frac{\Gamma(-z)\Gamma(-1 - \epsilon - z)\Gamma(-\epsilon - \delta - z)}{\Gamma(-\epsilon - z)} \frac{\Gamma(1 + z)\Gamma(1 - \epsilon + z)\Gamma(1 + \epsilon + \delta + z)}{\Gamma(2 + z)} \Big|_{z=-1-\epsilon}$$

$$= - \left(\frac{M^2}{Q^2} \right)^\epsilon e^{2\epsilon\gamma} \frac{\Gamma(1 - \epsilon)\Gamma(-\epsilon)\Gamma(\epsilon)}{\Gamma(1 - 2\epsilon)^2} \frac{\Gamma(1 + \epsilon)\Gamma(-\epsilon)\Gamma(-2\epsilon)}{\Gamma(1 - \epsilon)}$$

$$= \frac{1}{2\epsilon^4} - \frac{1}{2\epsilon^3} L + \frac{1}{4\epsilon^2} L^2 - \frac{1}{\epsilon} \left(\frac{1}{12} L^3 + \frac{4}{3} \zeta_3 \right) + \frac{1}{48} L^4 + \frac{4}{3} \zeta_3 L - \frac{\pi^4}{60} + \mathcal{O}(\epsilon)$$