PAUL SCHERRER INSTITUT

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# Evaluation of electroweak two-loop corrections in the high-energy limit

**Bernd Jantzen**

*Paul Scherrer Institut (PSI), CH-Villigen*

# Overview



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J.H. Kühn, A.A. Penin, *hep-ph/9906545*

J.H. Kühn, A.A. Penin, V.A. Smirnov, *Eur. Phys. J. C 17 (2000) 97*

J.H. Kühn, S. Moch, A.A. Penin, V.A. Smirnov, *Nucl. Phys. B 616 (2001) 286*

B. Feucht, J.H. Kühn, S. Moch, *Phys. Lett. B 561 (2003) 111*

B. Feucht, J.H. Kühn, A.A. Penin, V.A. Smirnov, *Phys. Rev. Lett. 93 (2004) 101802*

B. Jantzen, J.H. Kühn, A.A. Penin, V.A. Smirnov, *Phys. Rev. D 72 (2005) 051301(R)*

B. Jantzen, J.H. Kühn, A.A. Penin, V.A. Smirnov, *Nucl. Phys. B 731 (2005) 188*

**B. Jantzen, V.A. Smirnov, hep-ph/0603133 (→ Eur. Phys. J. C)**

# I Motivation

## Electroweak precision physics

- experimentally measured (LEP, Tevatron) at energy scales $\lesssim M_{W,Z}$
- upcoming accelerators (LHC, ILC) $\rightarrow$ TeV region
- new energy domain $\sqrt{s} \gg M_{W,Z}$ becomes accessible!

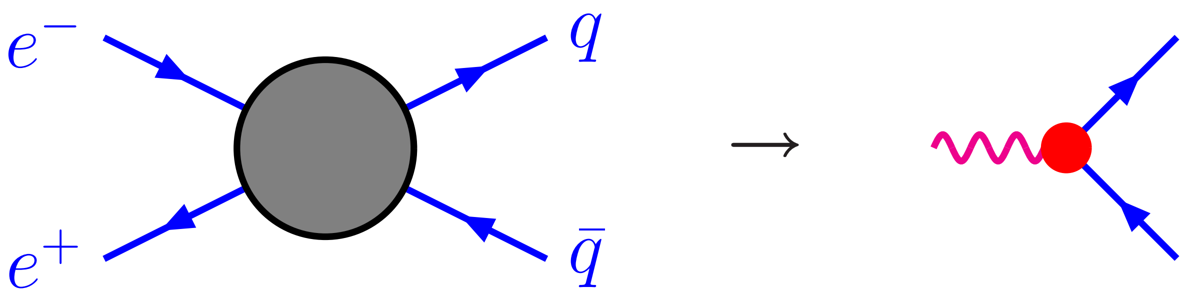
## Electroweak radiative corrections at high energies $\sqrt{s} \sim$ TeV $\gg M_{W,Z}$

Fadin et al. '00; Kühn et al. '00, '01, '05; Denner et al. '01, '03, '04, '05; Pozzorini '04; B.J. et al. '03, '04, '05, '06; ...

large corrections in *exclusive* cross sections

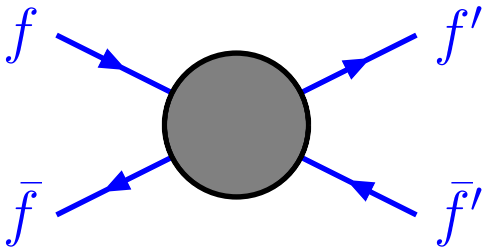
- electroweak corrections with large logarithms: $\alpha^n \ln^j(s/M_{W,Z}^2)$
- leading Sudakov logarithms ($j = 2n$, LL), but large coefficients in front of subleading logarithms ($j \leq 2n-1 \Rightarrow$ NLL, NNLL, N$^3$LL, ...)
- origin of Sudakov logarithms: mass singularities for $M_{W,Z} \rightarrow 0$
- 1-loop corrections $\sim 10\%$, 2-loop corrections $\sim 1\%$, needed for ILC
- individual logarithmic contributions even larger, but strong cancellations

# II Four-fermion scattering



**Four-fermion scattering:** $f\bar{f} \to f'\bar{f}'$, important class of processes

**Factorization of QED contributions:**



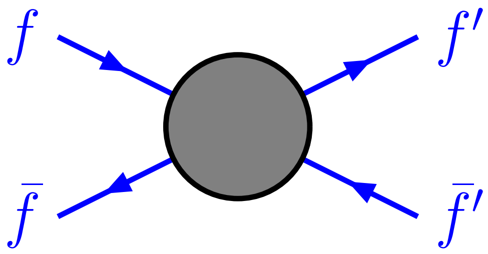

$$A = U_{\text{QED}} \cdot A_{\text{EW}}$$

- QED factor $U_{\text{QED}} \to$ IR singularities from virtual massless photons
  (regularized dimensionally or by small photon mass, compensated by real corrections)
- amplitude $A_{\text{EW}} \to$ remaining electroweak contributions, IR-safe
- calculate $A_{\text{EW}}$ by evaluating $A/U_{\text{QED}}$ with $M_{\text{photon}} = M_W$
  $\hookrightarrow$ works at NNLL accuracy ✓

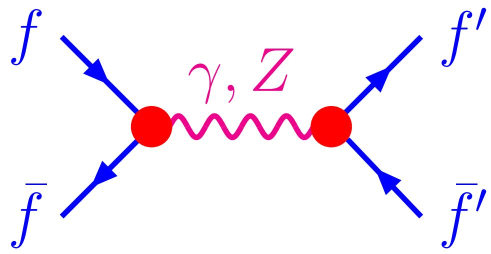
Problem at N$^3$LL (2-loop linear logarithm):
mixing of gauge groups $SU(2) \times U(1)_Y$ through Higgs mechanism
$\Rightarrow$ use simplified model without mixing: B.J., Kühn, Penin, Smirnov '04, '05

- factorization of QED contributions works at N$^3$LL accuracy
- single mass parameter: $M = M_W = M_{Z=W^3} = M_{\text{photon}=B}$
- include mass difference $(M_Z - M_W)$ by expansion around $M_Z \approx M_W$
- remaining error $\sim \mathcal{O}(\sin^2\theta_W) \sim 20\%$ in coefficient of linear 2-loop logarithm

## Factorization into form factor and reduced amplitude:

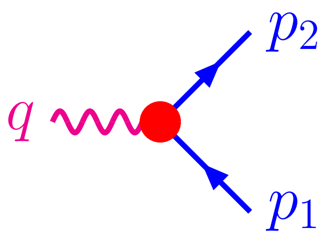



$$A_{\mathsf{EW}} = \frac{ig^2}{s} F^2 \tilde{A}$$




$\times \tilde{A}$

**Form factor** $F$ of vector current:




$$= F \cdot \bar{u}(p_2)\, \gamma^\mu\, u(p_1) + \mathcal{O}(\text{fermion masses})$$

**High-energy behaviour** $s \sim |t| \sim |u| \gg M_{W,Z}^2$ references: see Kühn et al. '01

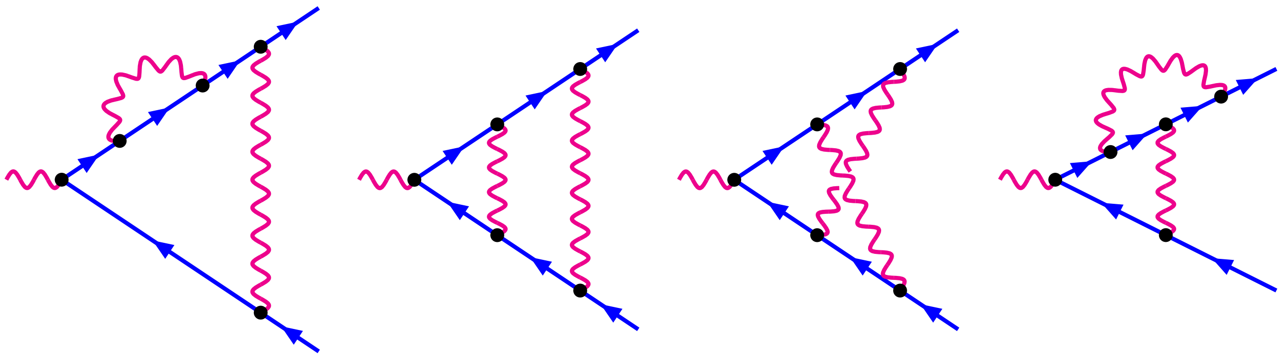
- all double logarithms $\alpha^n \ln^{2n} \rightsquigarrow$ form factors $F^2$
- *reduced amplitude* $\tilde{A} \longrightarrow$ only single logarithms $\alpha^n \ln^n$
- $\tilde{A}$ can be obtained from 1-loop and massless 2-loop calculations

$\Rightarrow$ For full logarithmic ($\mathrm{N^3LL}$) 2-loop amplitude: need form factor $F$

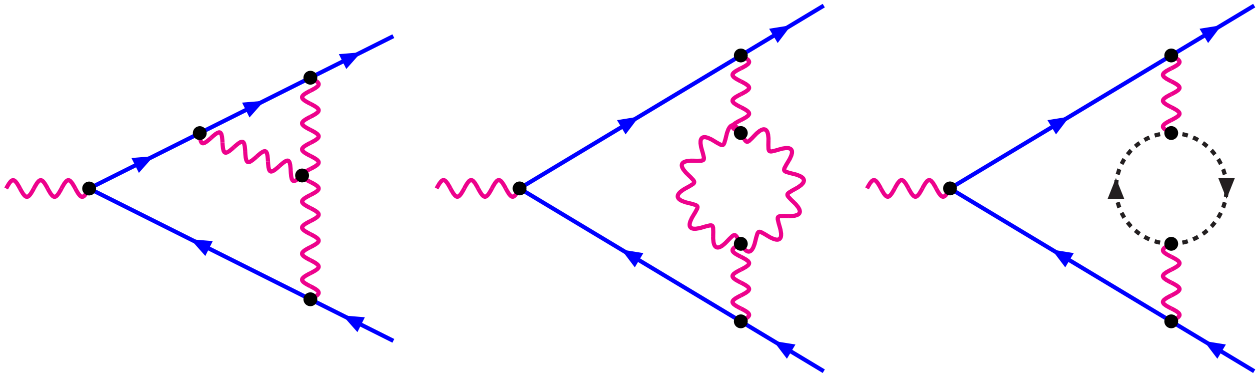
$\hookrightarrow$ evaluate 2-loop vertex diagrams

# SU(2) form factor in two loops: diagrams

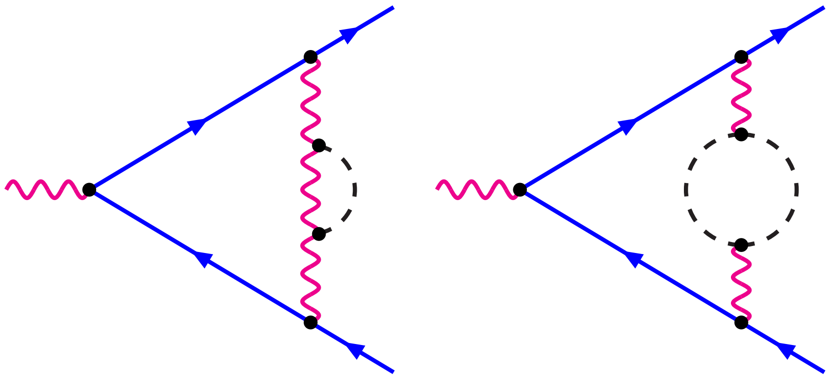
Abelian:
↪ U(1)



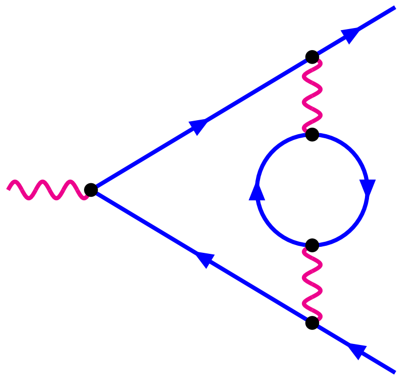
non-Abelian:



Higgs:



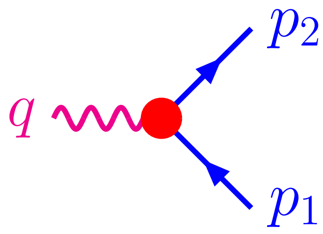
fermion:



+ 1-loop×1-loop corrections + renormalization

# High-energy behaviour of the form factor

↪ Sudakov limit:




$$= F(Q^2) \cdot \bar{u}(p_2)\, \gamma^\mu\, u(p_1)$$

- momentum transfer $-q^2 \equiv Q^2 \gg M^2 \equiv M_{W,Z}^2$

  $\left[\text{Euclidean } Q^2 > 0, \ \text{real } F \xrightarrow[\text{continuation}]{\text{analytic}} \text{Minkowskian } Q^2 = -s - i0 < 0\right]$

- neglect fermion masses → external on-shell fermions: $p_1^2 = p_2^2 = 0$

- *logarithmic approximation*: neglect terms suppressed by a factor of $M^2/Q^2$

  ↪ works well for 2-loop $n_f$ contribution where the exact result in $M^2/Q^2$ is known

  B.F., Kühn, Moch '03

  ⇒ contains powers of the large logarithm $\ln(Q^2/M^2)$

  ⇒ leading order of asymptotic expansion in $M^2/Q^2$

- only 2-loop logarithms $\ln^{4,3,2,1}$, non-logarithmic constant more difficult

- choose $M_{\text{Higgs}} = M_W$ → calculation easier, affects only N$^3$LL, small error

# SU(2) form factor in two loops: result

$$\left(\frac{\alpha}{4\pi}\right)^2 \Bigg[ \frac{9}{32}\ln^4\left(\frac{Q^2}{M^2}\right) - \frac{43}{48}\ln^3\left(\frac{Q^2}{M^2}\right) + \left(\frac{7}{8}\pi^2 - \frac{235}{48}\right)\ln^2\left(\frac{Q^2}{M^2}\right)$$ confirmed ✓

$$+ \left(\frac{13}{2}\sqrt{3}\,\mathrm{Cl}_2\left(\frac{\pi}{3}\right) + \frac{15}{4}\sqrt{3}\pi - \frac{61}{2}\zeta_3 - \frac{11}{24}\pi^2 + \frac{65}{4}\right)\ln\left(\frac{Q^2}{M^2}\right) \Bigg]$$ new!

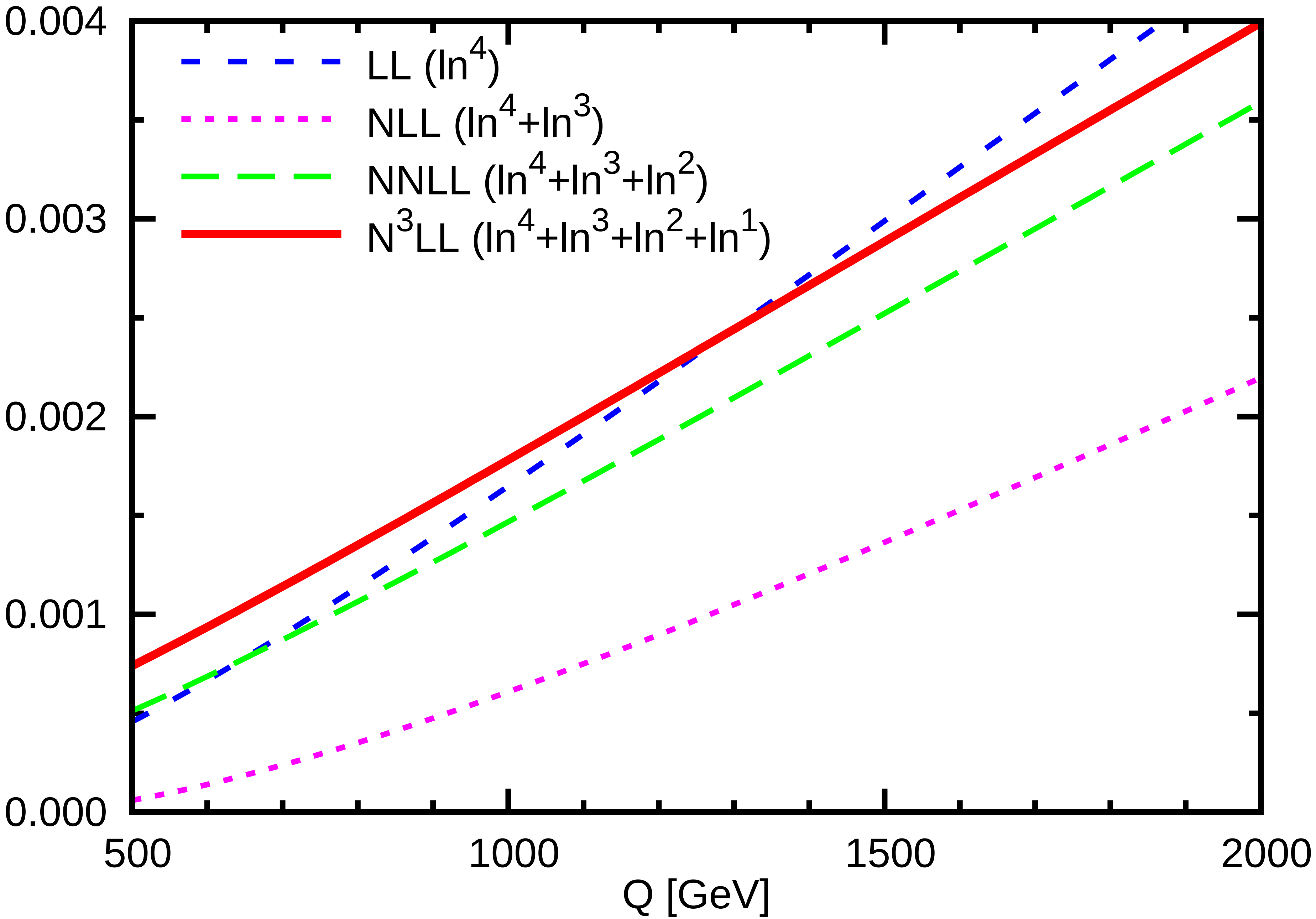
B.J., Kühn, Moch '03; B.J., Kühn, Penin, Smirnov '04, '05; B.J., Smirnov '06

$\ln^{4,3,2}$: Kühn, Moch, Penin, Smirnov '01

Sizes of logarithmic contributions (at $Q = 1\,\mathrm{TeV}$ in per mil):

| | | | | | |
|---|---|---|---|---|---|
| Abelian: | $+\,0.3\ln^4$ | $-\,1.7\ln^3$ | $+\,8.2\ln^2$ | $-\,11\ln$ | $+\,15$ |
| | $+1.6$ | $-2.0$ | $+1.9$ | $-0.5$ | $+0.1$ |
| fermionic: | | $-\,1.0\ln^3$ | $+\,9.5\ln^2$ | $-\,26\ln$ | $+\,42$ |
| | | $-1.2$ | $+2.2$ | $-1.2$ | $+0.4$ |
| non-Abelian + Higgs: | | $+\,1.8\ln^3$ | $-\,14\ln^2$ | $+\,43\ln$ | $-\,\ldots$ |
| | | $+2.1$ | $-3.2$ | $+2.0$ | |
| **total:** | $+\,0.3\ln^4$ | $-\,0.9\ln^3$ | $+\,3.7\ln^2$ | $+\,6.9\ln$ | |
| | $+1.6$ | $-1.0$ | $+0.9$ | $+0.3$ | |

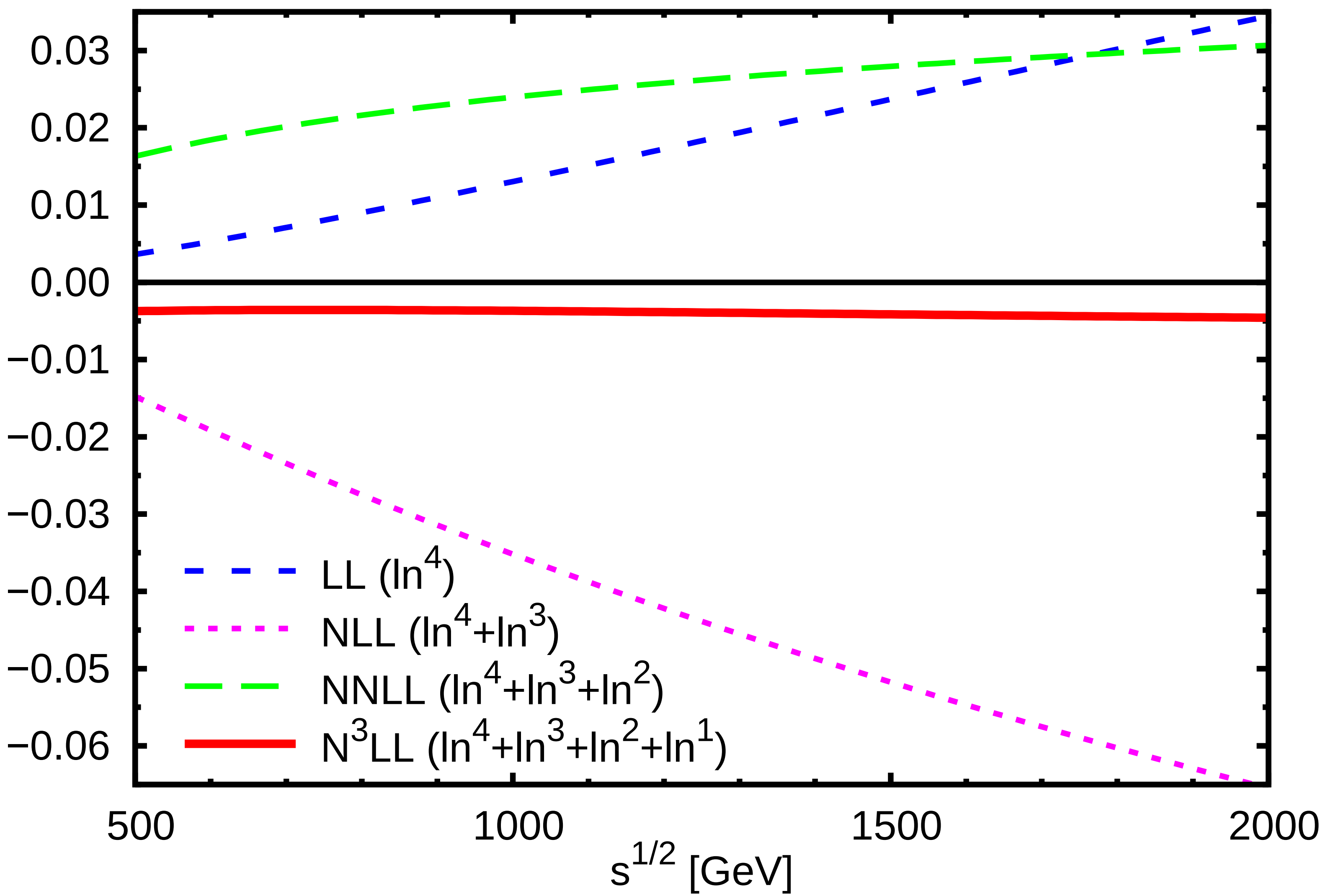
## SU(2) form factor in two loops: result (2)



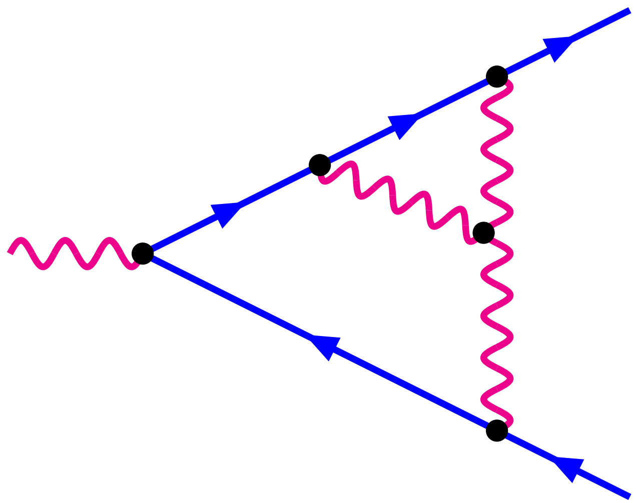
## Electroweak results: example $\sigma(e^+e^- \to q\bar{q})$ $(q = d, s)$

numerical 2-loop result:

$$\left(\frac{\alpha_{\text{ew}}}{4\pi}\right)^2 \left[ +2.79 \ln^4\left(\frac{s}{M_W^2}\right) - 51.98 \ln^3\left(\frac{s}{M_W^2}\right) + 321.34 \ln^2\left(\frac{s}{M_W^2}\right) - 757.35 \ln\left(\frac{s}{M_W^2}\right) \right]$$



# III Evaluating Feynman diagrams in the high-energy limit



$$= \,?\, + \mathcal{O}\left(\frac{M^2}{Q^2}\right)$$

## Reduction to scalar diagrams

- **given** from Feynman rules: $\mathcal{F}^\mu = \bar{u}(p_2)\,\Gamma^\mu(p_1,p_2)\,u(p_1)$
- **wanted:** form factor $F$ with $\mathcal{F}^\mu = F \cdot \bar{u}(p_2)\,\gamma^\mu\,u(p_1)$
- can be calculated using the properties of Dirac matrices and spinors, $\{\gamma^\mu,\gamma^\nu\} = 2g^{\mu\nu}$, $\not{p}_1\,u(p_1) = 0$, $\bar{u}(p_2)\,\not{p}_2 = 0$, combined with tensor reduction
- more elegantly with a *projector* on the form factor:

$$F = \frac{\mathrm{Tr}\left[\gamma_\mu\,\not{p}_2\,\Gamma^\mu(p_1,p_2)\,\not{p}_1\right]}{2(d-2)\,q^2}$$

- **output:** form factor $F$ in terms of *scalar Feynman integrals*

$$\int \mathrm{d}^d k_1 \int \mathrm{d}^d k_2\, \frac{\prod_{j=1}^{N} (\ell_j \cdot \ell'_j)^{\nu_j}}{\prod_{i=1}^{L} ({k'_i}^2 - M_i^2)^{n_i}}$$

with propagators and irreducible scalar products in the numerator

## Expansion by regions

Beneke, Smirnov '98

a powerful method for the asymptotic expansion of Feynman diagrams

- **given:** scalar Feynman integral & limit like $Q^2 \gg M^2$
- **wanted:** expansion of the *integral* in $M^2/Q^2$
- **problem:** direct expansion of the *integrand* leads to (new) IR/UV singularities

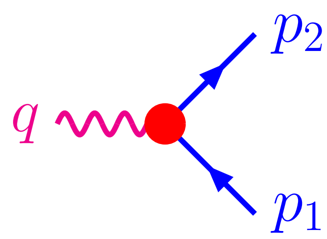
**Recipe for the method of expansion by regions:**

1. *divide* the integration domain into *regions* for the loop momenta
   (especially such regions where singularities are produced in the limit $M \to 0$)
2. in every region, *expand* the integrand in a *Taylor series* with respect to the parameters that are considered small *there*
3. *integrate* the expanded integrands over the *whole integration domain*
4. put to zero any *integral without scale* (like with dimensional regularization)

- usually only a few regions give non-vanishing contributions
- for logarithmic approximation: only leading order of the expansion needed
  $\hookrightarrow$ in step 2. all small parameters in the integrand are simply set to zero
- sometimes additional regularization (apart from $\varepsilon$) needed for individual regions

## Expansion by regions: example

### Vertex form factor in the Sudakov limit $Q^2 \gg M^2$




- typical regions for each loop momentum $k$:

| | | |
|---|---|---|
| hard | (h): | all components of $k \sim Q$ |
| soft | (s): | all components of $k \sim M$ |
| ultrasoft | (us): | all components of $k \sim M^2/Q$ |
| 1-collinear | (1c): | $k^2 \sim 2p_1 \cdot k \sim M^2, \quad 2p_2 \cdot k \sim Q^2$ |
| 2-collinear | (2c): | $k^2 \sim 2p_2 \cdot k \sim M^2, \quad 2p_1 \cdot k \sim Q^2$ |

- 1-loop vertex correction: $f = \dfrac{e^{\varepsilon\gamma_E}}{i\pi^{d/2}} \displaystyle\int \frac{\mathrm{d}^d k}{(k^2 - M^2)(k^2 - 2p_1 \cdot k)(k^2 - 2p_2 \cdot k)}$

$$f^{(h)} = \frac{1}{Q^2}\left[-\frac{1}{\varepsilon^2} + \frac{1}{\varepsilon}\ln(Q^2) - \frac{1}{2}\ln^2(Q^2) + \frac{\pi^2}{12} + \mathcal{O}\left(\frac{M^2}{Q^2}\right)\right]$$

$$f^{(1c)} + f^{(2c)} = \frac{1}{Q^2}\left[\frac{1}{\varepsilon^2} - \frac{1}{\varepsilon}\ln(Q^2) - \frac{1}{2}\ln^2(M^2) + \ln(M^2)\ln(Q^2) - \frac{5}{12}\pi^2 + \mathcal{O}\left(\frac{M^2}{Q^2}\right)\right]$$

$$\Rightarrow f = f^{(h)} + f^{(1c)} + f^{(2c)} = \frac{1}{Q^2}\left[-\frac{1}{2}\ln^2\left(\frac{Q^2}{M^2}\right) - \frac{\pi^2}{3} + \mathcal{O}\left(\frac{M^2}{Q^2}\right)\right]$$

# Expansion by regions: how its works

simple 1-dimensional example: $f = \int_0^\infty \frac{\mathrm{d}k\, k^{-\varepsilon}}{(k+m)(k+q)}\,, \quad m \ll q$

$$\left.\begin{array}{lll} \text{soft (s):} & k \sim m, & k < \Lambda \\ \text{hard (h):} & k \sim q, & k > \Lambda \end{array}\right\} \text{where } m \ll \Lambda \ll q$$

$$f = \int_0^\Lambda \frac{\mathrm{d}k\, k^{-\varepsilon}}{(k+m)(k+q)} + \int_\Lambda^\infty \frac{\mathrm{d}k\, k^{-\varepsilon}}{(k+m)(k+q)}$$

$$= \sum_{j=0}^\infty \frac{(-1)^j}{q^{j+1}} \int_0^\Lambda \frac{\mathrm{d}k\, k^{-\varepsilon+j}}{k+m} + \sum_{i=0}^\infty (-m)^i \int_\Lambda^\infty \frac{\mathrm{d}k\, k^{-\varepsilon-i-1}}{k+q}$$

$$= \sum_{j=0}^\infty \frac{(-1)^j}{q^{j+1}} \left( \int_0^\infty \frac{\mathrm{d}k\, k^{-\varepsilon+j}}{k+m} - \int_\Lambda^\infty \frac{\mathrm{d}k\, k^{-\varepsilon+j}}{k+m} \right) + \sum_{i=0}^\infty (-m)^i \left( \int_0^\infty \frac{\mathrm{d}k\, k^{-\varepsilon-i-1}}{k+q} - \int_0^\Lambda \frac{\mathrm{d}k\, k^{-\varepsilon-i-1}}{k+q} \right)$$

$$= \underbrace{\sum_{j=0}^\infty \frac{(-1)^j}{q^{j+1}} \int_0^\infty \frac{\mathrm{d}k\, k^{-\varepsilon+j}}{k+m}}_{f^{(s)}} + \underbrace{\sum_{i=0}^\infty (-m)^i \int_0^\infty \frac{\mathrm{d}k\, k^{-\varepsilon-i-1}}{k+q}}_{f^{(h)}} - \sum_{i=0}^\infty (-m)^i \sum_{j=0}^\infty \frac{(-1)^j}{q^{j+1}} \underbrace{\int_0^\infty \mathrm{d}k\, k^{-\varepsilon-i+j-1}}_{\to 0,\ \text{scaleless integral}}$$

$$= f^{(s)} + f^{(h)} = \frac{\Gamma(\varepsilon)\Gamma(1-\varepsilon)}{m^\varepsilon\, q} \sum_{j=0}^\infty \left(\frac{m}{q}\right)^j + \frac{\Gamma(-\varepsilon)\Gamma(1+\varepsilon)}{q^{1+\varepsilon}} \sum_{i=0}^\infty \left(\frac{m}{q}\right)^i$$

$$= \frac{\Gamma(\varepsilon)\Gamma(1-\varepsilon)}{(q-m)\, m^\varepsilon} + \frac{\Gamma(-\varepsilon)\Gamma(1+\varepsilon)}{(q-m)\, q^\varepsilon} = \frac{\ln(q/m)}{q-m} + \mathcal{O}(\varepsilon) \quad \checkmark$$

## Parameterization of Feynman integrals

- Schwinger parameters:

$$\frac{1}{A^n} = \frac{1}{i^n\,\Gamma(n)} \int_0^\infty \mathrm{d}\alpha\,\alpha^{n-1}\,e^{i\alpha A}\,, \quad \text{numerator } A^n = \left(\frac{1}{i}\frac{\partial}{\partial\alpha}\right)^n e^{i\alpha A}\bigg|_{\alpha=0}$$

⇒ any number of propagators and numerators may be combined

⇒ can always be transformed to (generalized) Feynman parameters

↪ evaluation:

$$\int \mathrm{d}^d k\, e^{i(\alpha k^2 + 2p\cdot k)} = i\pi^{d/2}\,(i\alpha)^{-d/2}\,e^{-ip^2/\alpha}$$

$$\int_0^\infty \frac{\mathrm{d}\alpha\,\alpha^{n-1}}{(A+\alpha B)^r} = \frac{\Gamma(n)\,\Gamma(r-n)}{\Gamma(r)\,A^{r-n}\,B^n}$$

- generalized Feynman parameters:

$$\prod_{i=1}^{L} \frac{1}{A_i^{n_i}} = \frac{\Gamma(\sum_i n_i)}{\prod_i \Gamma(n_i)} \left(\prod_i \int_0^\infty \mathrm{d}x_i\, x_i^{n_i-1}\right) \frac{\delta\Big(\sum_{j\in S} x_j - 1\Big)}{(\sum_i x_i A_i)^{\sum_i n_i}}\,, \qquad \emptyset \neq S \subseteq \{1,\dots,L\}$$

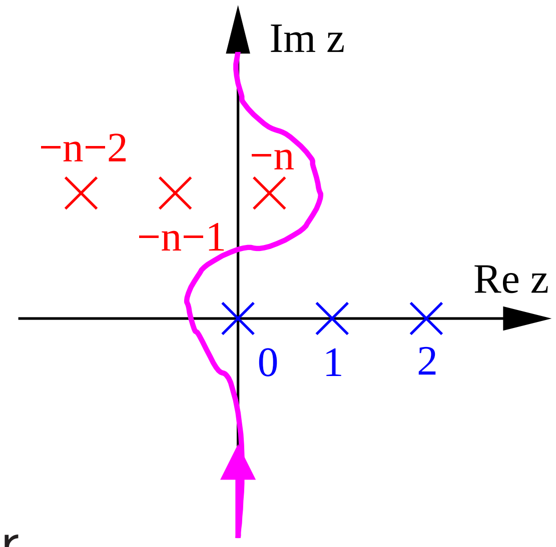
⇒ convenient also for non-standard propagators, e.g. $A_i = 2p\cdot k$

## Mellin–Barnes representation

Ussyukina '75; Boos, Davydychev '91

Feynman integrals with many parameters are hard to evaluate
↪ separate parameters by Mellin–Barnes representation:

$$\frac{1}{(A+B)^n} = \frac{1}{\Gamma(n)} \int_{-i\infty}^{i\infty} \frac{\mathrm{d}z}{2\pi i}\, \Gamma(-z)\, \Gamma(n+z)\, \frac{B^z}{A^{n+z}}$$




- Mellin–Barnes integrals go along the imaginary axis, leaving poles of $\Gamma(-z+\ldots)$ to the right and poles of $\Gamma(z+\ldots)$ to the left of the integration contour
- applicable to massive propagators $(A = k^2,\ B = -M^2)$ or to any complicated parameter integral
- evaluation: close the integration contour to the right $(|B| \le |A|)$ or to the left $(|B| \ge |A|)$ and pick up the residues within the contour: $\mathrm{Res}\,\Gamma(z)\big|_{z=-i} = (-1)^i/i!$
- applicable for asymptotic expansions: sum of residues yields expansion in powers of $(B/A)$ or $(A/B)$ and $\ln(A/B)$
- close link to *expansion by regions*:
  Mellin–Barnes representation of the full integral → sum of residues
  ↪ asymptotic expansion with contributions corresponding to the regions

## Mellin–Barnes integrals: extraction of singularities

$$I = \int_{-i\infty}^{i\infty} \frac{\mathrm{d}z}{2\pi i} \underbrace{\Gamma(\alpha - z)}_{\text{right poles}} \underbrace{\Gamma(-\alpha + \varepsilon + z)}_{\text{left poles}} f(z)$$

⇒ The right pole at $z = \alpha$ and the left pole at $z = \alpha - \varepsilon$ "glue together" for $\varepsilon \to 0$.
Close contour to the right:

$$-\mathrm{Res}\,\Gamma(\alpha - z)\Gamma(-\alpha + \varepsilon + z) f(z)\Big|_{z=\alpha} = \Gamma(\varepsilon) f(\alpha) = \frac{1}{\varepsilon} f(\alpha) + \mathcal{O}(\varepsilon^0)$$

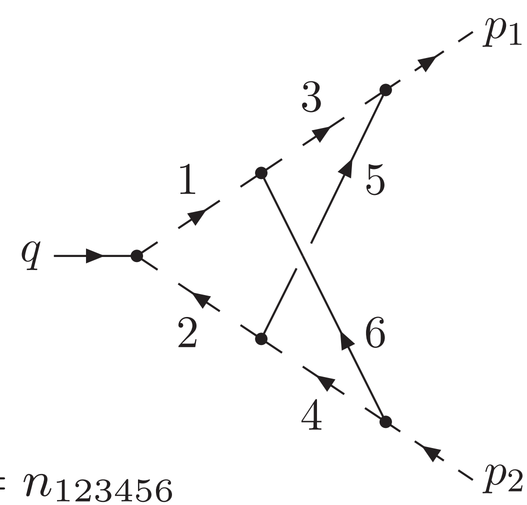
⇒ When a left pole and a right pole glue together, a singularity is produced!

Extraction of such singularities:

$$I = -\mathrm{Res}\,\Gamma(\alpha - z)\Gamma(-\alpha+\varepsilon+z) f(z)\Big|_{z=\alpha} + \int_{-i\infty}^{i\infty} \frac{\mathrm{d}z}{2\pi i} \underbrace{\Gamma(1 + \alpha - z)}_{\text{right poles}} \underbrace{\frac{\Gamma(-\alpha + \varepsilon + z)}{\alpha - z}}_{\text{left poles}} f(z)$$

Now the poles at $z = \alpha$ and $z = \alpha - \varepsilon$ both lie to the left of the integration contour.
↪ The integrand can safely be expanded in $\varepsilon$.

## Example: the non-planar vertex diagram




Scalar integrals with variable powers of propagators:

$$F_{\mathsf{NP}}(n_1,\ldots,n_7) = e^{2\varepsilon\gamma}\,(M^2)^{2\varepsilon}\,(Q^2)^{n-n_7-4}$$
$$\times \int\frac{\mathrm{d}^dk}{i\pi^{d/2}}\int\frac{\mathrm{d}^d\ell}{i\pi^{d/2}}\,\frac{(2k\cdot\ell)^{n_7}}{\big((p_1-k-\ell)^2\big)^{n_1}\,\big((p_2-k-\ell)^2\big)^{n_2}}$$
$$\times\frac{1}{(k^2-2p_1\cdot k)^{n_3}\,(\ell^2-2p_2\cdot\ell)^{n_4}\,(k^2-M^2)^{n_5}\,(\ell^2-M^2)^{n_6}}\,,\qquad n=n_{123456}$$

**Contributing regions:** (h-h), (1c-h), (h-2c), (1c-1c), (2c-2c), (1c-2c), (1c-1c'), (2c'-2c), (us'-us'), (1c-us'), (us'-2c).

**Leading term of (1c-h) region** $\iff$ $k^2\sim 2p_1\cdot k\sim M^2$, $2p_2\cdot k\sim Q^2$, $\ell\sim Q$:

$$F_{\mathsf{NP}}^{(\mathsf{1c\text{-}h})}(n_1,\ldots,n_7) = e^{2\varepsilon\gamma}\,(M^2)^{2\varepsilon}\,(Q^2)^{n-n_7-4}\int\frac{\mathrm{d}^dk}{i\pi^{d/2}}\int\frac{\mathrm{d}^d\ell}{i\pi^{d/2}}$$
$$\times\frac{\big((2p_2\cdot k)(2p_1\cdot\ell)/Q^2\big)^{n_7}}{\big(\ell^2-2p_1\cdot\ell+(2p_2\cdot k)(2p_1\cdot\ell)/Q^2\big)^{n_1}\,\big(\ell^2-2p_2\cdot(k+\ell)+(2p_2\cdot k)(2p_1\cdot\ell)/Q^2\big)^{n_2}}$$
$$\times\frac{1}{(k^2-2p_1\cdot k)^{n_3}\,(\ell^2-2p_2\cdot\ell)^{n_4}\,(k^2-M^2)^{n_5}\,(\ell^2)^{n_6}}+\mathcal{O}\Big(\frac{M^2}{Q^2}\Big)$$

## Example: (1c-h) region of the non-planar vertex diagram

Introduce Feynman or Schwinger parameters, integrate & transform into

$$F_{\text{NP}}^{(\text{1c-h})}(n_1,\dots,n_7) = \left(\frac{M^2}{Q^2}\right)^{2-n_{35}+\varepsilon} (-1)^n\, e^{2\varepsilon\gamma}\, \frac{\Gamma(\frac{d}{2}-n_{24})\Gamma(\frac{d}{2}-n_{16}+n_7)\Gamma(n_{35}-\frac{d}{2})}{\Gamma(d-n_{1246}+n_7)\,\prod_{i=1}^{6}\Gamma(n_i)}$$

$$\times \int_0^1 \mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3\, x_1^{n_1-1}(1-x_1)^{n_6-1}x_2^{n_2-1}(1-x_2)^{n_4-1}x_3^{n_{37}-1}(1-x_3)^{\frac{d}{2}-n_3-1}$$

$$\times \underbrace{\Gamma(n_{1246}-\tfrac{d}{2})\left[x_1(1-x_3)+x_2x_3\right]^{\frac{d}{2}-n_{1246}}}$$

Mellin–Barnes representation:

$$\int_{-i\infty}^{i\infty}\frac{\mathrm{d}z}{2\pi i}\,\Gamma(-z)\Gamma(n_{1246}-\tfrac{d}{2}+z)\,\big(x_1(1-x_3)\big)^z\,(x_2x_3)^{\frac{d}{2}-n_{1246}-z}$$

⇒ Expression with Mellin–Barnes integral:

$$F_{\text{NP}}^{(\text{1c-h})}(n_1,\dots,n_7) = \left(\frac{M^2}{Q^2}\right)^{2-n_{35}+\varepsilon} (-1)^n\, \frac{e^{2\varepsilon\gamma}\,\Gamma(\frac{d}{2}-n_{24})\Gamma(\frac{d}{2}-n_{16}+n_7)\Gamma(n_{35}-\frac{d}{2})}{\Gamma(n_1)\Gamma(n_2)\Gamma(n_3)\Gamma(n_5)\,\Gamma(d-n_{1246}+n_7)^2}$$

$$\times \int_{-i\infty}^{i\infty}\frac{\mathrm{d}z}{2\pi i}\,\frac{\Gamma(-z)\Gamma(\frac{d}{2}-n_{146}-z)\Gamma(\frac{d}{2}-n_{1246}+n_{37}-z)}{\Gamma(\frac{d}{2}-n_{16}-z)}$$

$$\times \frac{\Gamma(n_1+z)\Gamma(\frac{d}{2}-n_3+z)\Gamma(n_{1246}-\frac{d}{2}+z)}{\Gamma(n_{16}+z)}$$

## Example: evaluation of the (1c-h) region for special cases (1)

$$F_{\mathsf{NP}}^{(\mathsf{1c\text{-}h})}(1,1,1,0,1,0,0) = \left(\frac{M^2}{Q^2}\right)^{\varepsilon} e^{2\varepsilon\gamma} \frac{\Gamma(1-\varepsilon)^2\Gamma(\varepsilon)}{\Gamma(2-2\varepsilon)^2} \times \int_{-i\infty}^{i\infty} \frac{\mathrm{d}z}{2\pi i} \Gamma(-z)\Gamma(1-\varepsilon-z)\,\Gamma(1-\varepsilon+z)\Gamma(\varepsilon+z)$$

Solution known: 1st Barnes lemma

Barnes 1908

$$\int_{-i\infty}^{i\infty} \frac{\mathrm{d}z}{2\pi i} \Gamma(\alpha_1-z)\Gamma(\alpha_2-z)\,\Gamma(\alpha_3+z)\Gamma(\alpha_4+z) = \frac{\Gamma(\alpha_1+\alpha_3)\Gamma(\alpha_1+\alpha_4)\Gamma(\alpha_2+\alpha_3)\Gamma(\alpha_2+\alpha_4)}{\Gamma(\alpha_1+\alpha_2+\alpha_3+\alpha_4)}$$

$$\Rightarrow F_{\mathsf{NP}}^{(\mathsf{1c\text{-}h})}(1,1,1,0,1,0,0) = \left(\frac{M^2}{Q^2}\right)^{\varepsilon} e^{2\varepsilon\gamma} \frac{\Gamma(1-\varepsilon)^2\Gamma(\varepsilon)}{\Gamma(2-2\varepsilon)^2} \frac{\Gamma(1-\varepsilon)\Gamma(\varepsilon)\Gamma(2-2\varepsilon)}{\Gamma(2-\varepsilon)}$$

$$= \frac{1}{\varepsilon^2} + \frac{1}{\varepsilon}\Big(-\mathcal{L}+3\Big) + \frac{1}{2}\mathcal{L}^2 - 3\mathcal{L} + 7 + \mathcal{O}(\varepsilon)\,, \quad \mathcal{L} = \ln\left(\frac{Q^2}{M^2}\right)$$

# Example: evaluation of the (1c-h) region for special cases (2)

$$F_{\text{NP}}^{(\text{1c-h})}(1,1,1,0,1,1,0) = -\left(\frac{M^2}{Q^2}\right)^{\varepsilon} e^{2\varepsilon\gamma} \frac{\Gamma(1-\varepsilon)\Gamma(-\varepsilon)\Gamma(\varepsilon)}{\Gamma(1-2\varepsilon)^2} \times \int_{-i\infty}^{i\infty} \frac{\mathrm{d}z}{2\pi i} \Gamma(-z)\Gamma(-\varepsilon-z) \frac{\Gamma(1+z)\Gamma(1-\varepsilon+z)\Gamma(1+\varepsilon+z)}{\Gamma(2+z)}$$

- 1st possibility: expand integrand in $\varepsilon$ and/or cancel functions in denominator
  $\hookrightarrow$ transform to expressions solvable by Barnes lemma etc.
- 2nd possibility: close integration contour to the right and take residues directly:

$$\left(\frac{M^2}{Q^2}\right)^{\varepsilon} e^{2\varepsilon\gamma} \frac{\Gamma(1-\varepsilon)^3\Gamma(1+\varepsilon)^2}{\varepsilon^3\,\Gamma(1-2\varepsilon)^2} \sum_{i=0}^{\infty} \Bigg[ \underbrace{\frac{\Gamma(1-2\varepsilon+i)}{\Gamma(2-\varepsilon+i)}}_{\text{from } z=-\varepsilon+i} - \underbrace{\frac{\Gamma(1-\varepsilon+i)}{\Gamma(2+i)}}_{\text{from } z=i} \Bigg]$$

expand Gamma functions $\rightarrow$ sum up to (multiple) zeta values:

$$F_{\text{NP}}^{(\text{1c-h})}(1,1,1,0,1,1,0) = \frac{\pi^2}{6\varepsilon^2} + \frac{1}{\varepsilon}\left(-\frac{\pi^2}{6}\mathcal{L} + 2\zeta_3\right) + \frac{\pi^2}{12}\mathcal{L}^2 - 2\zeta_3\mathcal{L} + \frac{\pi^4}{40} + \mathcal{O}(\varepsilon)$$

# Example: evaluation of the (1c-h) region for special cases (3)

$$F_{\mathsf{NP}}^{(\mathsf{1c\text{-}h})}(1,\delta,1,1,1,1,0) = -\left(\frac{M^2}{Q^2}\right)^{\varepsilon} (-1)^{\delta}\, e^{2\varepsilon\gamma}\, \frac{\Gamma(1-\varepsilon-\delta)\Gamma(-\varepsilon)\Gamma(\varepsilon)}{\Gamma(\delta)\Gamma(1-2\varepsilon-\delta)^2}$$
$$\times \int_{-i\infty}^{i\infty} \frac{\mathrm{d}z}{2\pi i}\, \frac{\Gamma(-z)\Gamma(-1-\varepsilon-z)\Gamma(-\varepsilon-\delta-z)}{\Gamma(-\varepsilon-z)}\, \frac{\Gamma(1+z)\Gamma(1-\varepsilon+z)\Gamma(1+\varepsilon+\delta+z)}{\Gamma(2+z)}$$

Limit $\delta \to 0 \Rightarrow \dfrac{1}{\Gamma(\delta)} \to 0$, but gluing poles at $z = -1-\varepsilon$ and $z = -1-\varepsilon-\delta$:

$$F_{\mathsf{NP}}^{(\mathsf{1c\text{-}h})}(1,0,1,1,1,1,0) = \lim_{\delta\to 0} \left(\frac{M^2}{Q^2}\right)^{\varepsilon} (-1)^{\delta}\, e^{2\varepsilon\gamma}\, \frac{\Gamma(1-\varepsilon-\delta)\Gamma(-\varepsilon)\Gamma(\varepsilon)}{\Gamma(\delta)\Gamma(1-2\varepsilon-\delta)^2}$$
$$\times \operatorname{Res} \frac{\Gamma(-z)\Gamma(-1-\varepsilon-z)\Gamma(-\varepsilon-\delta-z)}{\Gamma(-\varepsilon-z)}\, \frac{\Gamma(1+z)\Gamma(1-\varepsilon+z)\Gamma(1+\varepsilon+\delta+z)}{\Gamma(2+z)}\Bigg|_{z=-1-\varepsilon}$$
$$= -\left(\frac{M^2}{Q^2}\right)^{\varepsilon} e^{2\varepsilon\gamma}\, \frac{\Gamma(1-\varepsilon)\Gamma(-\varepsilon)\Gamma(\varepsilon)}{\Gamma(1-2\varepsilon)^2}\, \frac{\Gamma(1+\varepsilon)\Gamma(-\varepsilon)\Gamma(-2\varepsilon)}{\Gamma(1-\varepsilon)}$$
$$= \frac{1}{2\varepsilon^4} - \frac{1}{2\varepsilon^3}\mathcal{L} + \frac{1}{4\varepsilon^2}\mathcal{L}^2 - \frac{1}{\varepsilon}\left(\frac{1}{12}\mathcal{L}^3 + \frac{4}{3}\zeta_3\right) + \frac{1}{48}\mathcal{L}^4 + \frac{4}{3}\zeta_3\mathcal{L} - \frac{\pi^4}{60} + \mathcal{O}(\varepsilon)$$

# IV Summary & outlook

## Four-fermion scattering

- calculation in the high-energy limit reduced to the SU(2) form factor
- 2-loop form factor calculated including all large logarithms
- electroweak 2-loop corrections obtained in $N^3$LL approximation

## Evaluating Feynman diagrams in the high-energy limit

- expansion by regions
- Mellin–Barnes representation

$\hookrightarrow$ effective combination of advanced methods

## Outlook: more legs & more scales

apply methods for Feynman diagrams with

- more than 3 external legs $\rightarrow$ depend on all $p_i \cdot p_j$
- different heavy masses $M_W$, $M_Z$, $M_{\text{Higgs}}$, $m_t$, ...

$\Rightarrow$ successful extraction of LL and NLL contribution from Mellin–Barnes integrals ✓